# Algebraic Geometry
# Lecture Notes

following the Part III course *Algebraic Geometry*
by Dr C. Birkar
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Cambridge

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These notes are unofficial lecture notes and are not endorsed by Dr. Birkar.

The current version of this text as well as the LaTeX source code can be found at `http://caramdir.at/math`.

# Contents



# Introduction

The content of these notes is mostly identical to the content of the course "Algebraic Geometry" held by Dr. C Birkar in Michaelmas term 2009 in Cambridge. Some parts have however been slightly reorganized. Further, I added some additional remarks and useful lemmas. These are marked by *. Notes in the margins usually refer to texts where additional information or omitted proofs can be found. Knowledge of commutative algebra will often be silently assumed in the text.

The document is split into three parts. The first one discusses topics that, while mostly developed for use in algebraic geometry, are used in other branches of mathematics as well. The second chapter will introduce the main objects of algebraic geometry – schemes and the morphisms between them. The final chapter discusses cohomological methods in algebraic geometry. I plan to add solutions to the example sheets in an appendix, but this is not done yet.

The course covered subsets of chapters II and III of Hartshorne's *Algebraic Geometry* [Har77]. Therefore that book serves as the basic reference for this text. Most of chapters 1 and 2 of these notes can also be found in Liu's *Algebraic Geometry and Arithmetic Curves* [Liu02], which sometimes provides a good second view on a topic. To get some more geometric intuition about schemes and some of the notions we will develop, I can highly recommend reading the examples provided in *The Geometry of Schemes* by Eisenbud and Harris [EH00].

If you find any error or have any comments how the text might be improved, please write me a mail at `me@caramdir.at`. Updated versions of this document can be found at `http://caramdir.at/math`.

## Preliminaries

**Conventions** All rings are commutative with one and $0 \neq 1$. All ring homomorphisms take 1 to 1. A graded ring is always of the form $S = \bigoplus_{d \geq 0} S_d$ while a graded module is of the form $M = \bigoplus_{d \in \mathbb{Z}} M_d$. A graded homomorphism of graded rings is a ring homomorphism that preserves degrees.

**Definition 0.1.** A homomorphism $f\colon A \to B$ of local rings $(A, \mathfrak{m}_A)$ to $(B, \mathfrak{m}_B)$ is a *local* homomorphism if $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

**Definition 0.2.** If $M = \bigoplus M_d$ is a graded module, we define $M(n)$ for $n \in \mathbb{Z}$ to be the graded module $M(n) = \bigoplus M(n)_d$ with $M(n)_d = M_{n+d}$. In particular, from a graded ring $S$ we get $S$-modules $S(n)$.

**Definition 0.3.** If $S$ is a graded ring, $M$ a graded $S$-module, $\mathfrak{p}$ a homogeneous prime ideal and $b \in S$ a homogeneous element, we write $S_{(\mathfrak{p})}$, $S_{(b)}$, $M_{(\mathfrak{p})}$, $M_{(b)}$ for the degree zero elements in $S_\mathfrak{p}$, $S_b$, $M_\mathfrak{p}$ and $M_b$ respectively.

In a expression containing a list of things, elements that are marked with a hat are omitted. For example, $t_1 \cdots \widehat{t_j} \cdots t_n$ means the product of the $t_1, \ldots, t_n$ except for $t_j$.

# 1. General Abstract Nonsense

Before beginning to study proper algebraic geometry, we will introduce some objects and ideas that – while they were primarily developed with algebraic geometry in mind – are really of more general use in mathematics. First will introduce sheaves, which essentially clarify any situation where there is a dichotomy of global versus local properties of a geometric space. Then we will discuss ringed spaces. These are an abstraction of what a "geometric space" is. Here I collected some definitions which make sense in this more general setting – listeners of the Part III Complex Manifolds lectures will note that the many definitions here were also made in the complex manifold class by simply replacing "ringed space" by "complex manifold". The finial section of this chapter will be devoted to the fundamentals of homological algebra.

## 1.1. Sheaves

**Definition 1.1.** Let $X$ be a topological space. A *presheaf* $\mathcal{F}$ on $X$ consists of the following data:

- For each open set $U \subseteq X$, an Abelian group $\mathcal{F}(U)$, the elements of which are called *sections over* $U$;
- For each inclusion of open sets $V \subseteq U \subseteq X$ a *restriction homomorphism* $\mathcal{F}(U) \to \mathcal{F}(V)$, $s \mapsto s|_V$.

Further, it has to fulfill the following conditions:

1. $\mathcal{F}(\emptyset) = 0$ and for each open $U$ the restriction map $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map;
2. If $W \subseteq V \subseteq U$ are open sets and $s \in (U)$, then $s|_W = (s|_V)_W$.

A presheaf $\mathcal{F}$ on $X$ is called a *sheaf* if it satisfies the *sheaf condition*:

3. For all open $U \subseteq X$: If $U = \bigcup U_i$ is an open cover and we are given $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j$, then there exists a *unique* $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all $i$.

*Example* 1.2. For any topological space $X$ setting

$$\mathcal{F}(U) = \{s \colon U \to \mathbb{R} \text{ continuous}\}$$

with the usual restriction of functions gives a sheaf $\mathcal{F}$.

**Definition 1.3.** If $U$ is an open subset of $X$ and $\mathcal{F}$ is a (pre)sheaf on $X$, then $\mathcal{F}|_U$ is the (pre)sheaf on $U$ given by $V \mapsto \mathcal{F}(V)$ for $V \subseteq U$ open. It is called the *restriction* of $\mathcal{F}$ to $U$.

**Lemma* 1.4.** *Let $\{U_i\}$ be family of open subsets on $X$ and set $U = \bigcup U_i$. Let $U_{ij} = U_i \cap U_j$. For any presheaf $\mathcal{F}$ on $X$ there is a complex of Abelian groups* [Liu02, lemma 2.2.7]

$$0 \to \mathcal{F}(U) \xrightarrow{d_0} \prod_i \mathcal{F}(U_i) \xrightarrow{d_1} \prod_{i,j} \mathcal{F}(U_{ij}),$$

*where $d_0 \colon s \mapsto (s|_{U_i})_i$ and $d_1 \colon (s_i)_i \mapsto \big(s_i|_{U_{ij}} - s_j|_{U_{ij}}\big)_{i,j}$.*

*Then $\mathcal{F}$ is a sheaf if and only if this complex is exact for every family of open subsets of $X$.*

*Proof.* This is just a reformulation of the sheaf condition. □

**Definition 1.5.** The *stalk* $\mathcal{F}_x$ of a (pre)sheaf $\mathcal{F}$ at a point $x \in X$ is

$$\mathcal{F}_x = \varinjlim_{\substack{U \text{ open}, \\ x \in U}} \mathcal{F}(U).$$

*Remark* 1.6. Every element of $\mathcal{F}_x$ is represented by a pair $(U, s)$, where $x \in U$ open and $s \in \mathcal{F}(U)$. Two such elements $(U, s)$, $(V, t)$ are equal in $\mathcal{F}_x$ if there is an open set $W \subseteq U \cap V$ with $s|_W = t|_V$.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\phi_V"] & \mathcal{G}(V) \end{tikzcd}$$

**Definition 1.7.** Let $\mathcal{F}$, $\mathcal{G}$ be (pre)sheafs on $X$. A *morphism of (pre)sheaves*$\phi\colon \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that $\phi$ commutes with the restriction maps: for every inclusion of open sets $U \subseteq V$, we have $\phi_U(\cdot)|_V = \phi_V(\cdot|_V)$. A morphism of (pre)sheaves is an *isomorphism* if it has a two-sided inverse that is also a morphism.

*Remark* 1.8. For each $x \in X$, $\phi\colon \mathcal{F} \to \mathcal{G}$ induces a homomorphism $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ by $(U, s) \mapsto (U, \phi_U(s))$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, dashed, "\psi"] \\ & \mathcal{G} \end{tikzcd}$$

**Definition 1.9.** Let $\mathcal{F}$ be a presheaf on a topological space $F$. A sheaf $\mathcal{F}^+$ together with a morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{F}^+$ is called the *sheaf associated to* $\mathcal{F}$ if the following universal property holds: For every sheaf $\mathcal{G}$ and every morphism of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ there exists a unique morphism of sheaves $\psi\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\phi = \psi \circ \alpha$.

**Theorem 1.10.** *Let $\mathcal{F}$ be a presheaf on a topological space $X$. Then the associated sheaf $(\mathcal{F}^+, \alpha)$ exists and is unique up to unique isomorphism.*

*Proof.* Uniqueness follows directly from the universal property.

Let $\mathcal{F}^+(U)$ be the set

$$\left\{ s\colon U \to \coprod_{x \in U} \mathcal{F}_x : \begin{array}{l} \forall x \in U : s(x) \in \mathcal{F}_x \text{ and there is } x \in V \subseteq U \text{ open} \\ \text{and } t \in \mathcal{F}(V) \text{ such that } \forall y \in V : s(y) = (V, t) \in \mathcal{F}_y \end{array} \right\}.$$

(NB: This means that the sections of $\mathcal{F}^+$ are locally sections of $\mathcal{F}$.) The natural restriction maps make $\mathcal{F}^+$ a sheaf. For $U \subseteq X$ and $s \in \mathcal{F}(U)$, let $\alpha_U(s) \in \mathcal{F}^+(U)$ be the function $x \mapsto (U, s) \in \mathcal{F}_x$.

Now suppose we are given $\phi\colon \mathcal{F} \to \mathcal{G}$. We need to construct $\psi\colon \mathcal{F}^+ \to \mathcal{G}$. Let $s \in \mathcal{F}^+(U)$. By construction there exists an open cover $U = \bigcup U_i$ and $s_i \in \mathcal{F}(U_i)$ such that $\alpha_{U_i}(s_i) = s|_{U_i}$. Let $t_i = \phi(s_i)$. We easily see that $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. Therefore, since $\mathcal{G}$ is a sheaf, there exists a *unique* $t \in \mathcal{G}(U)$ such that $t|_{U_i} = t_i$ and we have to put $\psi_U(s) = t$. □

*Remark* 1.11. For each $x \in X$, the natural homomorphism $\mathcal{F}_x \to \mathcal{F}_x^+$ is an isomorphism.

**Definition 1.12.** Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. The *presheaf kernel* $\ker \phi$ of $\phi$ is given by $(\ker \phi)(U) = \ker \phi_U$. The *presheaf image* $\operatorname{im} \phi^{pre}$ of $\phi$ is given by $(\operatorname{im} \phi^{pre})(U) = \operatorname{im} \phi_U$.

Now, let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then $\ker \phi$ is a sheaf, called the *kernel* of $\phi$. The *image* $\operatorname{im} \phi$ of $\phi$ is the sheaf associated to $\operatorname{im} \phi^{pre}$. (Note that it can be seen as a subsheaf of $\mathcal{G}$.) The morphism $\phi$ is *injective* if $\ker \phi = 0$ and *surjective* if $\operatorname{im} \phi = \mathcal{G}$.

*Remark* 1.13. Even if $\mathcal{F} \to \mathcal{G}$ is a surjective morphism, the maps $\mathcal{F}(U) \to \mathcal{G}(U)$ are in general not surjective (see Exercise A.1).

**Theorem 1.14.** *Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on $X$. Then $\phi$ is injective, surjective or an isomorphism if and only if for each $x \in X$, $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ has this property.*

*Proof.*

- *Injectivity*: First suppose that $\phi$ is injective and pick any $x \in X$. We have to show that $\phi_x$ is injective. Suppose we have $s \in \mathcal{F}(U)$ (with $x \in U$) such that $\phi_x(U, s) = (U, \phi_U(s)) = 0 \in \mathcal{G}_x$. By definition this means that there exists $W \subseteq U$ with $x \in W$ such that

  $$0 = \phi_U(s)|_W = \phi_W(s|_W).$$

  Hence $s|_W \in \ker \phi_W$. Since $\phi$ is injective, $s|_W = 0$, so $(U, s) = (W, s|_W) = 0 \in \mathcal{F}_x$. Thus $\phi_x$ is injective.

  Now suppose that $\phi_x$ is injective for all $x$. Let $s \in \mathcal{F}(U)$ with $\phi_U(s) = 0$. Then $\phi_x(U, s) = (U, \phi_U(s)) = 0 \in \mathcal{G}_x$ for all $x$ in $U$. Since $\phi_x$ is injective $(U, s) = 0$ in $\mathcal{F}_x$. In other words, for all $x \in U$ there exists an open neighborhood $W \subseteq U$ of $x$ with $s|_W = 0$. By the sheaf condition $s = 0$ in $\mathcal{F}(U)$, so $\phi_U$ is injective.

- *Surjectivity*: Suppose that $\phi$ is surjective and pick any $x \in X$. Let $(U, t)$ be any element in $\mathcal{G}_x$. Since $\phi$ is surjective, there exists an open neighborhood $W \subseteq U$ of $x$ and $s \in \mathcal{F}(W)$ with $\phi_W(s) = t|_W$. Hence $\phi_x(W, s) = (W, \phi_W(s)) = (W, t|_W) = (U, s) \in \mathcal{G}_x$, i.e. $\phi_x$ is surjective.

  Conversely suppose all $\phi_x$ are surjective and pick $t \in \mathcal{G}(U)$. For all $x \in U$, by the surjectivity of $\phi_x$, there exists an open neighborhood $W \subseteq U$ of $x$ and $s \in \mathcal{F}(W)$ with $\phi_x(W, s) = (U, t) \in \mathcal{G}_x$. So for all $x \in U$, there exists an open neighborhood $V \subseteq W$ of $x$ such that $\phi_W(s)|_V = t_V$. In other words, sections of $\mathcal{G}$ are locally the image of sections of $\mathcal{F}$. Hence $\operatorname{im} \mathcal{F} = \mathcal{G}$.

- A morphism $\phi$ is an isomorphism if and only if all $\phi_U$ are bijective.

  If $\phi$ is an isomorphism, then we have just shown that all $\phi_x$ are bijective and hence isomorphisms.

  Suppose that all $\phi_x$ are isomorphisms. We have already shown that all $\phi_U$ are injective. Let $t \in \mathcal{G}(U)$. As above, for all $x \in U$ there exists a neighborhood $V \subseteq U$ of $x$ and $s_V \in \mathcal{F}(V)$ with $\phi_V(s_V) = t|_V$. By the injectivity of $\phi$, the sections $s_V$ are uniquely determined and are compatible. Hence the sheaf condition gives $s \in \mathcal{F}(U)$ with $\phi_U(s) = t$, i.e. $\phi_U$ is surjective. □

**Definition 1.15.** A *sequence of sheaves* is a family $\mathcal{F}_i$ $(i \in \mathbb{Z})$ of sheaves on $X$ and morphisms $\phi_i \colon \mathcal{F}_i \to \mathcal{F}_{i+1}$. It is usually written in the form

$$\cdots \longrightarrow \mathcal{F}_{-2} \xrightarrow{\phi_{-2}} \mathcal{F}_{-1} \xrightarrow{\phi_{-1}} \mathcal{F}_0 \xrightarrow{\phi_0} \mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \longrightarrow \cdots.$$

A *complex of sheaves* is a sequence with $\operatorname{im} \phi_i \subseteq \ker \phi_{i+1}$ for all $i \in \mathbb{Z}$. It is an *exact sequence* if $\operatorname{im} \phi_i = \ker \phi_{i+1}$ for all $i \in \mathbb{Z}$. A *short exact sequence* is an exact sequence of the form

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0. \tag{1.1}$$

*Remark* 1.16. A sequence of the form (1.1) is exact if and only if

$$0 \to (\mathcal{F}_1)_x \to (\mathcal{F}_2)_x \to (\mathcal{F}_3)_x \to 0$$

is exact for all $x \in X$.

*Example* 1.17. Let $X$ be a topological space and $A$ a fixed Abelian group. Define a presheaf $\mathcal{F}^{pre}$ on $X$ by $\mathcal{F}^{pre}(U) = A$ for all $U \neq \emptyset$ with the identities as restriction maps. Then $\mathcal{F} = (\mathcal{F}^{pre})^+$ is called the *constant sheaf* defined by $A$ on $X$. It is isomorphic to the sheaf $\mathcal{G}$ given by $\mathcal{G}(U) = \{s \colon U \to A \text{ locally constant}\}$.

**Definition 1.18.** Assume that $f \colon X \to Y$ is a continuous map of topological spaces and $\mathcal{F}$ is a sheaf on $X$. The *direct image sheaf* (or *push-down*) $f_*\mathcal{F}$ is the sheaf on $Y$ given by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)) \qquad U \subseteq X \text{ open}.$$

*Remark* 1.19. Let $X = \{y\} \subseteq Y$ and $f\colon X \hookrightarrow Y$. A sheaf $\mathcal{F}$ on $X$ is is completely determined by $\mathcal{F}(X) = A$. The direct image sheaf is given by

$$(f_*\mathcal{F})(U) = \begin{cases} A, & y \in U \\ 0, & \text{otherwise} \end{cases}.$$

It is called the *skyscraper sheaf* on $Y$ at $y$ given by $A$. Note that $(f_*\mathcal{F})_{y'} = 0$ for $y' \notin \overline{\{y\}}$.

**Definition 1.20.** Suppose $f\colon X \to Y$ is a continuous map of topological spaces and $\mathcal{G}$ is a sheaf on $Y$. Then the sheaf on $X$ associated to the presheaf

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

is called the *inverse image* of $\mathcal{G}$ and is denoted $f^{-1}\mathcal{G}$.

*Remark** 1.21. For every $x \in X$, $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$. Also, if $V$ is an open subset of $Y$ and $\iota\colon V \to Y$ is the injection, then $\iota^{-1}\mathcal{G} = \mathcal{G}|_V$.

[Har77, Exercise II.1.18]

*Remark** 1.22. There are natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$.

[EH00, Proposition I-12]

**Lemma* 1.23.** *Let $\mathcal{B}$ be a base of the topological space $X$ and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $X$. Given a collection of maps $\widetilde{\phi}_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ for all $U \in \mathcal{B}$ that commute with the restriction maps, there is a unique morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ of sheaves such that $\phi_U = \widetilde{\phi}_U$ for all $U \in \mathcal{B}$.*

## 1.2. Ringed Spaces

**Definition 1.24.** A *ringed space* $(X, \mathcal{O}_X)$ is a topological space $X$ together with a sheaf of rings, called the *structure sheaf*, $\mathcal{O}_X$ on $X$. A *morphism of ringed spaces* $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ consists of a continuous map $f\colon Y \to X$ and a morphism of sheaves $f^\#\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that all stalks $(\mathcal{O}_X)_P$, $P \in X$, are local rings. A *morphism of locally ringed spaces* $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism in the above sense such that all induced homomorphisms $(\mathcal{O}_X)_{f(Q)} \to (\mathcal{O}_Y)_Q$, $Q \in Y$ are local homomorphisms.

In both cases, an isomorphism is a morphism with a two-sided inverse.

*Remark** 1.25. The notation $f^\#$ was not used in the lecture, but is used in many books (like [Har77, Liu02, EH00]). If the space $X$ is clear, we will sometimes just write $\mathcal{O}$ for $\mathcal{O}_X$. For the stalk $(\mathcal{O}_X)_x$ $(x \in X)$ we will usually write $\mathcal{O}_{X,x}$ and sometimes just $\mathcal{O}_x$.

**Definition 1.26.** Let $(X, \mathcal{O})$ be a locally ringed space and $x \in X$. The maximal ideal of the local ring $\mathcal{O}_x$ is denoted $\mathfrak{m}_x$. The *residue field* at $x$ is $k(x) = \mathcal{O}_x/\mathfrak{m}_x$.

**Definition 1.27.** Let $(X, \mathcal{O}_X)$ be a ringed space. An *$\mathcal{O}_X$-module* is a sheaf $\mathcal{F}$ on $X$ such that for every open $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and such that for every inclusion of open sets $V \subseteq U$ in $X$ and $s \in \mathcal{O}_X(U)$, $m \in \mathcal{F}(U)$, $(sm)|_V = s_V m_V$. A *morphism of $\mathcal{O}_X$-modules* $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that for every open $U \subseteq X$ the map $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism. The set of all $\mathcal{O}_X$-module morphisms from $\mathcal{F}$ to $\mathcal{G}$ is denoted by $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

*Remark* 1.28. The direct sum, direct limit and inverse limit of $\mathcal{O}_X$-modules are again $\mathcal{O}_X$-modules. The kernel and image of a map of $\mathcal{O}_X$-modules are $\mathcal{O}_X$-modules. If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces and $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module (via $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$).

**Definition 1.29.** If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, we define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is morphism of ringed spaces and we have an $\mathcal{O}_Y$-module $\mathcal{G}$, then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module, but not an $\mathcal{O}_X$-module. The idea is to tensor it with $\mathcal{O}_X$ to obtain an $\mathcal{O}_X$-module. For this note that we have a morphism of sheaves $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$, which induces a morphism $f^{-1}\mathcal{O}_Y \to f^{-1}f_*\mathcal{O}_X$. We also have a natural morphism $f^{-1}f_*\mathcal{O}_X \to \mathcal{O}_X$ (see remark 1.22). Together they give a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

**Definition 1.30.** Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. The *inverse image* of $\mathcal{G}$ is

$$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

**Definition 1.31.** Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{L}$ is called *invertible*, if for every point $x$ of $X$, there exists an open neighborhood $U$ of $x$ with $\mathcal{L}|_U = \mathcal{O}_X|_U$.

**Theorem 1.32.** *The set of isomorphism classes of invertible sheaves on a ringed space $X$ forms a group under $\otimes_{\mathcal{O}_X}$. The neutral element is $\mathcal{O}_X$ and the inverse of an invertible sheaf $\mathcal{L}$ is given by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.*

**Definition 1.33.** The group of isomorphism classes of invertible sheaves over a ringed space $X$ is called the *Picard group* and denoted by $\mathrm{Pic}(X)$.

*Proof of theorem 1.32.* Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on $(X, \mathcal{O}_X)$. For every $x \in X$ there exists an open neighborhood $U$ of $x$ such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$ and $\mathcal{M}|_U \cong \mathcal{O}_X|_U$. So $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M})|_U \cong \mathcal{O}_X|_U \otimes_{\mathcal{O}_X|_U} \mathcal{O}_X|_U \cong \mathcal{O}_X|_U$. Thus the product of two invertible sheaves is again invertible.

Since $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{L}$ for every invertible sheaf $\mathcal{L}$, the neutral element has to be $\mathcal{O}_X$.

We will now define an isomorphism $\phi\colon \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$ as the associated morphism of sheaves to the presheaf morphism given for each open subset $W \subseteq X$ by

$$\begin{aligned} \mathcal{L}(W) \otimes_{\mathcal{O}_X(W)} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)(W) &\to \mathcal{O}_X(W) \\ l \otimes \vartheta &\mapsto \vartheta_W(l) \end{aligned}$$

Pick $x \in X$ and a neighborhood $U$ of $x$ such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)|_U \cong \mathcal{O}_X$ (in particular this means that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is invertible) and $\phi_U$ is an isomorphism. This can be done for each $x \in X$, so $\phi$ is an isomorphism. □

## 1.3. Derived Functors

No proofs are given in this section. References can be found in [Har77, Section III.1].

**Definition 1.34.** An *Abelian category* is a category which behaves like the category of Abelian groups.

*Example* 1.35. The following categories are Abelian:

- The category of Abelian groups, denoted $\mathfrak{Ab}$.
- The category of modules over a ring $R$.
- The category of sheaves of Abelian groups on a topological space $X$, denoted $\mathfrak{Sh}(X)$.
- The category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$, denoted $\mathfrak{M}(X)$.

**Definition 1.36.** Let $\mathfrak{A}$ be an Abelian category.

- A *complex* $A^\bullet$ in $\mathfrak{A}$ is a sequence
$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots,$$
where the $A^i$ are objects in $\mathfrak{A}$ and the $d^i$ are morphisms in $\mathfrak{A}$ with $d^{i+1} \circ d^i = 0$ for all $i$.
- The *$i$-th cohomology of* $A^\bullet$ is $h^i(A^\bullet) = \ker d^i / \operatorname{im} d^{i-1}$.
- A *morphism* of complexes $f\colon A^\bullet \to B^\bullet$ is a commutative diagram
$$\begin{tikzcd} \cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d, "f^{i-1}"] & A^i \arrow[r] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\ \cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots \end{tikzcd}$$
- A *short exact sequence* of complexes
$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$
is a sequence of complexes such that all induced sequences
$$0 \to A^i \to B^i \to C^i \to 0$$
are exact.

**Theorem 1.37.** *Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence of complexes. Then there exists natural maps that give a* long exact sequence of cohomology
$$\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to \cdots.$$

**Definition 1.38.** Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a (covariant) functor between Abelian categories.

- $F$ is *additive* if the maps : $\operatorname{Hom}(A, A') \xrightarrow{F} \operatorname{Hom}(FA, FA')$ are homomorphisms of Abelian groups.
- $F$ is called *left exact* if it is additive and takes every short exact sequence
$$0 \to A' \to A \to A'' \to 0$$
in $\mathfrak{A}$ to an exact sequence
$$0 \to FA' \to FA \to FA''$$
in $\mathfrak{B}$.

**Definition 1.39.**

- An object $I$ in an Abelian category $\mathfrak{A}$ is called *injective* if for every commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & A' \arrow[dl, dashed, "\exists g"] \\ & I & \end{tikzcd}$$
in $\mathfrak{A}$ with an exact first row there exists a morphism $g\colon A' \to I$ which fits into the diagram.

- An *injective resolution* of an object $A$ of an Abelian category $\mathfrak{A}$ is an exact sequence
$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots,$$
such that all $I^j$ are injective in $\mathfrak{A}$.
- An Abelian category is said to have *enough injectives* is every object has an injective resolution.

**Definition 1.40.** Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a left exact functor between Abelian categories and assume that $\mathfrak{A}$ has enough injectives. For each $i \in \mathbb{N}$, define the *right derived functor* $R^iF\colon \mathfrak{A} \to \mathfrak{B}$ *of* $F$ in the following way: For each object $A$ of $\mathfrak{A}$ pick an injective resolution
$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$
and apply $F$ to get a complex
$$D^\bullet\colon 0 \to FI^0 \to FI^1 \to FI^2 \to \cdots.$$
Then $RF^i(A) = h^i(D^\bullet)$.

**Theorem 1.41.** *With the notation of the above definition the following statements hold:*

1. *$R^iF(A)$ does not depend on the choice of injective resolution (up to natural isomorphism).*
2. *$R^0F \cong F$.*
3. *Each short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathfrak{A}$ induces (in a canonical way) a long exact sequence*
$$\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \to R^{i+1}F(A') \to \cdots$$
*in $\mathfrak{B}$.*
4. *Each commutative diagram*
$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$
*with exact rows induces a commutative diagram*
$$\begin{tikzcd} \cdots \arrow[r] & R^iF(A') \arrow[r] \arrow[d] & R^iF(A) \arrow[r] \arrow[d] & R^iF(A'') \arrow[r] \arrow[d] & R^{i+1}F(A') \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & R^iF(B') \arrow[r] & R^iF(B) \arrow[r] & R^iF(B'') \arrow[r] & R^{i+1}F(B') \arrow[r] & \cdots \end{tikzcd}$$
*with exact rows.*

**Definition 1.42.** With the notation above: An object $J \in \mathfrak{A}$ is called *$F$-acyclic* if $R^iF(J) = 0$ for all $i \geq 1$.

**Theorem 1.43.** *Again using the above notation, if*
$$0 \to A \to J^0 \to J^1 \to J^2 \to \cdots$$

*is an exact sequence such that all $J^i$ are $F$-acyclic (this is called an* acyclic resolution *of $A$) and*

$$E^\bullet \colon 0 \to FJ^0 \to FJ^1 \to FJ^2 \to \cdots$$

*is the corresponding complex in $\mathfrak{B}$, then $RF^i(A) = h^i(E^\bullet)$ for all $i \geq 0$.*

# 2. Schemes

In this chapter we will begin the study of schemes – the main objects of algebraic geometry – and the morphisms between them. The first section will introduce affine schemes, which are the building blocks of general schemes which we will introduce next. As a special case we will discuss projective space. Then we will take a look at some special classes of morphisms between schemes. Already in the first section we will introduce the sheaf of modules on an affine scheme associated to a module. In section 5 we will study sheaves of modules more closely. We will finish this chapter by giving short introductions to the notions of divisors and differentials.

## 2.1. Affine Schemes

**Definition 2.1.** Let $A$ be a ring. Then $\operatorname{Spec} A$ is the set of prime ideals of $A$. If $\mathfrak{a}$ is a ideal of $A$ we define $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{a} \subseteq \mathfrak{p}\}$.

**Lemma 2.2.** *Let $A$ be a ring and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}_i$ be ideals of $A$.*

1. $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$
2. $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$
3. $V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \Leftrightarrow \sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$

*Proof.*

1. For a prime ideal $\mathfrak{p}$:
$$\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b}) \Leftrightarrow \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p} \Leftrightarrow \mathfrak{a} \subseteq \mathfrak{p} \text{ or } \mathfrak{b} \subseteq \mathfrak{p} \Leftrightarrow \mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b}).$$
A similar argument works for $V(\mathfrak{a} \cap \mathfrak{b})$.
2. Again for $\mathfrak{p}$ prime:
$$\mathfrak{p} \in V\left(\sum \mathfrak{a}_i\right) \Leftrightarrow \sum \mathfrak{a}_i \subseteq \mathfrak{p} \Leftrightarrow \forall i : \mathfrak{a}_i \subseteq \mathfrak{p} \Leftrightarrow \forall i : \mathfrak{p} \in V(\mathfrak{a}_i) \Leftrightarrow \mathfrak{p} \in \bigcap V(\mathfrak{a}_i).$$
3. Follows immediately from
$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}.$$
□

**Definition 2.3.** Taking the sets of the form $V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$ to be the closed sets defines a topology in $\operatorname{Spec} A$, called the *Zariski topology*. The sets $D(f) = \operatorname{Spec} A \setminus V((f))$, $f \in A$ are called *principal open sets*.

*Remark* 2.4. The principal open sets form a basis for the Zariski topology: Let $U = \operatorname{Spec} A \setminus V(\mathfrak{a})$ be open. If $\mathfrak{a} = (b_i : i \in I)$, then $V(\mathfrak{a}) = \bigcap_{i \in I} V((b_i))$ and hence $U = \bigcup D(b_i)$.

**Lemma* 2.5.** *Let $A$ be a ring. Then all principal open sets are quasi-compact. In particular* $\operatorname{Spec} A$ *is quasi-compact.*

*Proof.* Let $D(b)$ be covered by open sets $U_i$. Since the principal open sets form a basis of the topology we may without loss of generality assume that all $U_i$ are principal open sets contained in $D(b)$; say $U_i = D(d_i)$.

From $D(b) = \bigcup D(d_i)$, we obtain $V((b)) = \bigcap V((d_i)) = V(\sum(d_i))$, so $\sqrt{(b)} = \sqrt{\sum(d_i)}$. Hence there exists $l \geq 1$ such that $b^l \in \sum(d_i)$. In particular, there exist finitely many $d_i$, say $d_1, \ldots, d_n$, and $b_i \in A$ with $b^l = \sum_{i=1}^n b_i d_i$. Therefore $V(b) \supseteq V(\sum_{j=1}^n (d_j)) \supseteq V(\sum(d_i)) = V(b)$. Hence we have equality and $D(b) = \bigcup_{i=1}^n D(d_i)$. □

**Definition 2.6.** Let $A$ be a ring, $X = \operatorname{Spec}(A)$ and $M$ an $A$-module. We will define the *sheaf associated to* $M$, denoted $\widetilde{M}$, of $A$-modules on $X$: For an open subset $U \subseteq X$, define $\widetilde{M}(U)$ to be the set of functions

$$s\colon U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}}$$

such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ and such that $s$ is locally a fraction $\frac{m}{b}$ with $m \in M$, $b \in A$, i.e. for each $\mathfrak{p} \in U$ there exist a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $m \in M$ and $b \in A$ such that for all $\mathfrak{q} \in V$, $b \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{b}$ in $A_{\mathfrak{q}}$.

**Definition 2.7.** Let $A$ be a ring, $X = \operatorname{Spec} A$. We make $X$ into a ringed space by taking the structure sheaf to be $\mathcal{O}_X = \widetilde{A}$.

*Remark* 2.8. Since this definition is so fundamental, we will write it out: For an open $U \subseteq X$, define $\mathcal{O}_X(U)$ to be the set of functions

$$s\colon U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ and such that $s$ is locally a quotient of elements in $A$, i.e. for each $\mathfrak{p} \in U$ there exist a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $a, b \in A$ such that for all $\mathfrak{q} \in V$, $b \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{b}$ in $A_{\mathfrak{q}}$. Then $\mathcal{O}_X$ with the obvious restriction maps (which are ring homomorphisms) is a sheaf on $X$ making $(X, \mathcal{O}_X)$ into a ringed space.

**Theorem 2.9.** *Let $A$ be a ring, $X = \operatorname{Spec} A$ and $M$ and $A$-module. Then:*

1. *$\widetilde{M}$ is an $\mathcal{O}_X$-module.*
2. *For each $\mathfrak{p} \in X$, $\left(\widetilde{M}\right)_{\mathfrak{p}} = M_{\mathfrak{p}}$.*
3. *For each $b \in A$, $\widetilde{M}(D(b)) = M_b$.*
4. *$\widetilde{M}(X) = M$.*

*Proof.*

1. As $M_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$-module this is immediately obvious.
2. We define a homomorphism of $A_{\mathfrak{p}}$-modules $f\colon \left(\widetilde{M}\right)_{\mathfrak{p}} \to M_{\mathfrak{p}}$ by $(U, s) \mapsto s(\mathfrak{p}) \in M_{\mathfrak{p}}$. By construction of $\widetilde{M}$, this is well-defined.

   First, we will show that $f$ is injective: Let $f((U, s)) = 0$ and let $s$ be locally given by $\frac{m}{b}$, $m \in M$, $b \in A$. In particular this means that $\frac{m}{b} = 0$ in $M_{\mathfrak{p}}$. So there exists $c \in A \setminus \mathfrak{p}$ with $cm = 0$. Thus $\frac{m}{b} = 0$ at every point of $D(b) \cap D(c) \ni \mathfrak{p}$. Therefore the restriction of $s$ to $U \cap D(b) \cap D(c) \neq \emptyset$ is 0 and so $(U, s) = 0$.

   Now we will show that $f$ is surjective: Pick any element $\frac{m}{b} \in M_{\mathfrak{p}}$. Let $W = D(b) \ni \mathfrak{p}$ and take $s \in \widetilde{M}(W)$ to be the "constant function" $s\colon W \to \coprod_{\mathfrak{q} \in W} M_{\mathfrak{q}} \colon \mathfrak{q} \mapsto \frac{m}{b} \in M_{\mathfrak{q}}$. Then $f((W, s)) = \frac{m}{b} \in M_{\mathfrak{p}}$.

3. We define a function $g\colon M_b \to \widetilde{M}(D(b))$ by sending $\frac{m}{b^n}$ to the "constant function" defined by $\frac{m}{b^n}$ (as above).

   The function $g$ is well-defined: If $\frac{m}{b^n} = \frac{m'}{b^{n'}}$ in $M_b$, then $b^l(mb^{n'} - m'b^n) = 0$ in $M$ for some $l$ and hence $\frac{m}{b^n} = \frac{m'}{b^{n'}} \in M_{\mathfrak{q}}$ for every $\mathfrak{q}$ in $D(b)$.

   First we show that $g$ is injective. Assume that $g(\frac{a}{b^n}) = 0$ in $\widetilde{M}(D(b))$. This means that $\frac{a}{b^n} = 0 \in M_{\mathfrak{q}}$ for each $\mathfrak{q} \in D(b)$. Hence there are elements $h^{(\mathfrak{q})} \in A \setminus \mathfrak{q}$ such that $h^{(\mathfrak{q})}m = 0$. Let $\mathfrak{a}$ be annihilator of $m$. Then $h^{(\mathfrak{q})} \in \mathfrak{a}$, but $h^{(\mathfrak{q})} \notin \mathfrak{q}$. So $\mathfrak{a} \nsubseteq \mathfrak{q}$ for all $\mathfrak{q} \in D(b)$. Therefore $V(\mathfrak{a}) \cap D(b) = \emptyset$, i.e. $V(\mathfrak{a}) \subseteq V(b)$. By lemma 2.2, $b \in \sqrt{\mathfrak{a}}$, so $b^l \in \mathfrak{a}$ for some $l \geq 1$. Thus $b^l m = 0$ and $\frac{m}{b^n} = 0$ in $M_b$.

   Proving the surjectivity of $g$ is harder. Let $s \in \widetilde{M}(D(b))$. By definition of $\widetilde{M}$ there exists an open cover $\{U_i\}$ of $D(b)$ such that $s|_{U_i}$ is given by $\frac{m_i}{e_i}$. Since the principal open sets are a base of the topology we may assume that $U_i = D(d_i)$ for some $d_i \in A$. Principal open subsets are quasi-compact (Lemma 2.5), so we may assume that the cover is finite.

   For the fractions to be well-defined we must have $D(d_i) \subseteq D(e_i)$, i.e. $\sqrt{(d_i)} \subseteq \sqrt{(e_i)}$. So there exist $n_i \geq 1$ and $e_i' \in A$ with $d_i^{n_i} = e_i' e_i$. This implies $\frac{m_i}{e_i} = \frac{e_i' m_i}{d_i^{n_i}}$. The upshot of this is that by replacing $d_i$ by $d_i^{n_i}$ (since $D(d_i) = D(d_i^{n_i})$) and $m_i$ by $e_i' m_i$ we may assume that $D(b)$ is covered by open subsets $U_i = D(d_i)$ such that $s|_{U_i}$ is given by $\frac{m_i}{d_i}$.

   We will change the elements $d_i$ such that the compatibility conditions become simpler. Notice that $D(d_i) \cap D(d_j) = D(d_i d_j)$. So by construction, $\frac{m_i}{d_i}|_{D(d_i d_j)} = \frac{m_j}{d_j}|_{D(d_i d_j)}$, i.e. $\frac{m_i}{d_i} = \frac{m_j}{d_j}$ in $M_{d_i d_j}$. So there exists $n_{ij} \geq 1$ with $(d_i d_j)^{n_{ij}}(m_i d_j - m_j d_i) = 0$ in $M$. Since there are only finitely many possibilities to pick the pair $(i, j)$, we can take $n'$ to be larger than all of the $n_{ij}$. Rewriting the equality condition, we have $d_j^{n'+1}(d_i^{n'} m_i) - d_i^{n'+1}(d_j^{n'} m_j) = 0$. By replacing $d_i$ by $d_i^{n'+1}$ and $m_i$ by $d_i^{n'} m_i$ we can assume that $d_i m_j - d_j m_i = 0$ while still having $s$ represented as $\frac{m_i}{d_i}$ on $D(d_i)$.

   From $D(b) = \bigcup D(d_i)$ we obtain $\sqrt{(b)} = \sqrt{\sum (d_i)}$ and hence $b^l = \sum_{i=1}^n b_i d_i$ for some $l \in \mathbb{N}$ and $b_i \in A$. Set $m = \sum_{i=1}^n b_i m_i$. Then

   $$d_j m = \sum b_i d_j m_i = \sum b_i d_i m_j = \left(\sum b_i d_i\right) m_j = b^l m_j.$$

   Therefore $\frac{m}{b^l} = \frac{m_j}{d_j}$ on $D(d_j)$. Hence $g(\frac{m}{b^l}) = s$ everywhere. So $g$ is surjective.

4. Obtained from the previous point with $b = 1$. □

**Corollary 2.10.** *Let $A$ be a ring, $X = \operatorname{Spec}(A)$. Then:*

1. *For each $\mathfrak{p} \in X$, $\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$.*
2. *For each $b \in A$, $\mathcal{O}_X(D(b)) = A_b$.*
3. *$\mathcal{O}_X(X) = A$.*

*Proof.* This is just the special case $M = A$ of theorem 2.9. □

In particular the corollary asserts that $(X, \mathcal{O}_X)$ is a locally ringed space. Actually it is a very important one, giving rise to the following definition:

**Definition 2.11.** An *affine scheme* is a locally ringed space that is isomorphic to $(\operatorname{Spec} A, \widetilde{A})$ for some ring $A$.

We usually write $\operatorname{Spec} A$ both for the topological space and the affine scheme.

**Theorem 2.12.** *Let $\alpha\colon A \to B$ be a homomorphism of rings. Write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then $\alpha$ induces (naturally) a morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed spaces. Every morphism (of locally ringed spaces) $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ comes from a homomorphism $A \to B$.*

*Proof.* Given $\alpha\colon A \to B$, we define $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ by $f(\mathfrak{p}) = \alpha^{-1}(\mathfrak{p})$. Since $f^{-1}(V(\mathfrak{a})) = V(\alpha(\mathfrak{a})B)$, $f$ is continuous. We need to define $f^{\#}\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$. Let $f(\mathfrak{p}) = \mathfrak{q}$. Then we have a local homomorphism $\alpha_{\mathfrak{p}}\colon A_{\mathfrak{q}} \to B_{\mathfrak{p}}$. By definition, each element $s \in \mathcal{O}_Y(U)$ is a functions $U \to \coprod_{\mathfrak{q}\in U} A_{\mathfrak{q}}$. Define $f_U^{\#}(s) = t \in \mathcal{O}_Y(f^{-1}(U))$ as $t\colon f^{-1}U \to \coprod_{\mathfrak{p}\in f^{-1}U} B_{\mathfrak{p}}$ with $t(\mathfrak{p}) = \alpha_{\mathfrak{p}}(s(f(\mathfrak{p})))$. If $s$ is locally given by a quotient $\frac{a}{b}$, then $t$ is locally given by the quotient $\frac{\alpha(a)}{\alpha(b)}$. So $t$ is indeed an element of $f_*\mathcal{O}_Y$. The induced maps on the stalks $(\mathcal{O}_X)_{\mathfrak{q}} \to (f_*\mathcal{O}_Y)_{\mathfrak{q}} \to (\mathcal{O}_Y)_{\mathfrak{p}}$ are just the local homomorphisms $\alpha_{\mathfrak{q}}$, so $(f, f^{\#})$ is indeed a morphism of locally ringed spaces.

Conversely let $(f, f^{\#})$ be a morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$. By 2.10.3 we have a homomorphism of rings $\alpha = f_X^{\#}\colon A \to B$. We will show that $\alpha$ induces the given morphism. For any $\mathfrak{p} \in Y$, put $\mathfrak{q} = f(\mathfrak{p})$. Then we have a commutative diagram

$$\begin{array}{ccc} A = \mathcal{O}_X(X) & \xrightarrow{\alpha} & \mathcal{O}_Y(Y) = B \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} \\ A_{\mathfrak{q}} = (\mathcal{O}_X)_{\mathfrak{q}} & \xrightarrow{f_{\mathfrak{p}}^{\#}} & (\mathcal{O}_Y)_{\mathfrak{p}} = B_{\mathfrak{p}} \end{array}$$

From this we obtain:

$$\mathfrak{q} = \beta^{-1}(\mathfrak{q}_{\mathfrak{q}}) = \beta^{-1}((f_{\mathfrak{p}}^{\#})^{-1}(\mathfrak{p}_{\mathfrak{p}})) = \alpha^{-1}(\gamma^{-1}(\mathfrak{p}_{\mathfrak{p}})) = \alpha^{-1}(\mathfrak{p}).$$

Hence $f = \alpha^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$. Also $f_{\mathfrak{p}}^{\#}$ is given by $\alpha_{\mathfrak{p}}$. Now, for any open set $U \subseteq X$ with $\mathfrak{q} \in U$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{f_U^{\#}} & \mathcal{O}_Y(f^{-1}(U)) \\ \Big\downarrow & & \Big\downarrow \\ A_{\mathfrak{q}} = (\mathcal{O}_X)_{\mathfrak{q}} & \xrightarrow{f_{\mathfrak{p}}^{\#} = \alpha_{\mathfrak{p}}} & (\mathcal{O}_Y)_{\mathfrak{p}} = B_{\mathfrak{p}} \end{array}$$

A section $s \in \mathcal{O}_X(U)$ is determined by all values $s(\mathfrak{q})$, $\mathfrak{q} \in U$. But $f_{\mathfrak{p}}^{\#}(s(\mathfrak{q})) = \alpha_{\mathfrak{p}}(s(\mathfrak{q}))$. So $f_U^{\#}(s)$ is determined by the values $\alpha_{\mathfrak{p}}(s(\mathfrak{q}))$, hence $f^{\#}$ coincides with the map defined in the first step. $\square$

## 2.2. Schemes

**Definition 2.13.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $x \in X$ has an open neighborhood $U$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. An *(iso)morphism of schemes* is an (iso)morphism of locally ringed spaces.

*Examples* 2.14. Let $A$ be a ring and $X = \operatorname{Spec} A$.

- If $A$ is a field, then $X$ has only one point.
- If $A$ is a DVR, then $X$ has only two points: $\mathfrak{o}$ and $\mathfrak{m}_A$ (the maximal ideal of $A$). The latter is a closed point (i.e. $\{\mathfrak{m}_A\}$ is a closed subset of $X$), while $\mathfrak{o}$ is not a closed point. Actually it is dense, i.e. $\overline{\{\mathfrak{o}\}} = X$ ($\mathfrak{o}$ is a generic point).

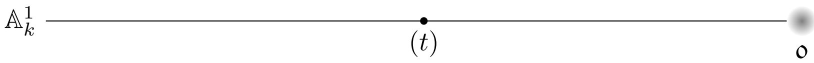
**Definition 2.15.** Let $A$ be a ring. The affine scheme $\operatorname{Spec}(A[t_1, \dots, t_n])$ is called *affine $n$-space* over $A$ and is denoted $\mathbb{A}_A^n$.

*Remark* 2.16. If $k$ is an algebraically closed field, then $\mathbb{A}_k^n$ is essentially the classical $n$-dimensional affine space. Notice that the new affine space $\mathbb{A}_k^n$ has more points than the classical one. E.g., the points of $\mathbb{A}_k^1$ are the prime ideals $(t-a)$, $a \in k$ plus the zero ideal. *see also* [EH00, section II.1.1]

*Example* 2.17. The space $X = \operatorname{Spec}\mathbb{Z}$ consists of the prime ideals $(p)$ with $p \in \mathbb{Z}$ a prime number together with $\mathfrak{o}$. We have $(\mathcal{O}_X)_{\mathfrak{o}} = \mathbb{Z}_{\mathfrak{o}} = \mathbb{Q}$ and $(\mathcal{O}_X)_{(p)} = \mathbb{Z}_{(p)}$ and the residue field $k((p)) = \mathbb{F}_p$.

*Example* 2.18. For any ring $A$ and ideal $\mathfrak{a} \lhd A[t_1, \dots, t_n]$, we have a scheme $\operatorname{Spec}(A[t_1, \dots, t_n]/\mathfrak{a})$. In particular take $A = k$ to be an algebraically closed field. Classically, two ideals $\mathfrak{a}, \mathfrak{b}$ define the same subset of affine space if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. However, as schemes, $\operatorname{Spec}(A[t_1, \dots, t_n]/\mathfrak{a})$ is not necessarily isomorphic to $\operatorname{Spec}(A[t_1, \dots, t_n]/\mathfrak{b})$.

For instance, if $\mathfrak{a} = (t) \lhd k[t]$, $\mathfrak{b} = (t^m) \lhd k[t]$, then $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, but the two spectra differ. In an informal sense $\operatorname{Spec}(k[t]/\mathfrak{b})$ corresponds to the point $(t) \in \mathbb{A}_k^1$ with multiplicity $m$. *see also* [EH00, section II.3.1]




*Example* 2.19. Take the ideal $\mathfrak{a} = (t_1^2 + t_2^2 + 1)$ of $\mathbb{R}[t_1, t_2]$. Classically, the affine set $V(\mathfrak{a})$ is empty. However the scheme $X = \operatorname{Spec}(\mathbb{R}[t_1, t_2]/\mathfrak{a})$ is not empty. Also we have a homomorphism $\mathbb{R}[t_1, t_2]/\mathfrak{a} \to \mathbb{C}[t_1, t_2]/\mathfrak{a}$ which induces a morphism of schemes $\operatorname{Spec}(\mathbb{C}[t_1, t_2]/\mathfrak{a}) \to \operatorname{Spec}(\mathbb{R}[t_1, t_2]/\mathfrak{a})$. *see also* [EH00, section II.2]

**Definition 2.20.** Let $(X, \mathcal{O}_X)$ be a scheme.

- $X$ is *reduced*, if $\mathcal{O}_X(U)$ is reduced (i.e. has no nilpotent elements) for all open $U \subseteq X$.
- $X$ is *irreducible*, if $X$ is irreducible as a topological space (i.e. if $X = X_1 \cup X_2$ for closed $X_i$, then $X_1 = X$ or $X_2 = X$ or equivalently if no two open subsets of $X$ are disjoint).
- $X$ is *integral*, if $\mathcal{O}_X(U)$ is an integral domain for all $U \subseteq X$.

*Example* 2.21. Let $X = \operatorname{Spec}(A)$. Then:

- $X$ is reduced if and only if $A$ is reduced (i.e. $\operatorname{Nil}(A) = 0$).
- $X$ is irreducible if and only if $\operatorname{Nil}(A)$ is a prime ideal. [$X$ is irreducible $\Leftrightarrow$ if $X = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{ab})$, then $X = V(\mathfrak{a})$ or $X = V(\mathfrak{b})$ $\Leftrightarrow$ if $\mathfrak{ab} \subseteq \sqrt{\mathfrak{o}}$, then $\mathfrak{a} \subseteq \sqrt{\mathfrak{o}}$ or $\mathfrak{b} \subseteq \sqrt{\mathfrak{o}}$ $\Leftrightarrow$ $\sqrt{\mathfrak{o}} = \operatorname{Nil}(A)$ is prime.]
- $X$ is reduced and irreducible, if and only if $\operatorname{Nil}(A) = 0$ is a prime ideal, which is the case if an only if $A$ is an integral domain.

**Lemma 2.22.** *Let $X$ be a scheme and $s \in \mathcal{O}_X$. Then the set*

$$W_s = \{x \in X : s \notin \mathfrak{m}_x \subseteq (\mathcal{O}_X)_x\}$$

*is an open subset of $X$.*

*Proof.* By covering $X$ with open affine sets, we can assume that $X$ is affine. Let $X = \operatorname{Spec} A$ and pick $x = \mathfrak{p} \in X$. As $\mathfrak{m}_x$ is the localization of $\mathfrak{p}$, we have $s \in \mathfrak{m}_x \subseteq (\mathcal{O}_X)_x = A_{\mathfrak{p}}$ if and only if $s \in \mathfrak{p}$ (where we write $s$ both for the section and its image in the stalk). This in turn holds if and only if $\mathfrak{p} \in V((s))$. Therefore $X \setminus W_s$ is closed, so $W_s$ is open. $\square$

**Theorem 2.23.** *A scheme is integral if and only if it is reduced and irreducible.*

*Proof.* Let $(X, \mathcal{O}_X)$ be the scheme.

First assume that $X$ is integral. So by definition $\mathcal{O}_X(U)$ is an integral domain for each open set $U$. As integral domains are reduced rings, $X$ must be reduced. We will now show that $X$ is also irreducible. Suppose that this was not true, i.e. that there exist proper closed subsets $X_1, X_2$ of $X$ with $X = X_1 \cup X_2$. Then $(X \setminus X_1) \cap (X \setminus X_2) = \emptyset$, where $X \setminus X_i$ are nonempty open sets. Let $U_i \subseteq X \setminus X_i$ be open affine sets. Since $U_1 \cap U_2 = \emptyset$, the sheaf condition implies that $\mathcal{O}_X(U_1 \cup U_2) \cong \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ which is not an integral domain. This is a contradiction to the assumption that $X$ is integral, so $X$ must be irreducible.

Now assume that $X$ is reduced and irreducible. We have to show that $X$ is integral. Let $U$ be an open subset of $X$ and suppose we have elements $a, b \in \mathcal{O}_X(U)$ with $ab = 0$ in $\mathcal{O}_X(U)$. By the lemma, $W_a, W_b \subseteq U$ are open sets.

Let $V \subseteq U$ be an open affine set. Since both reduced and irreducible are properties that are inherited by $V$, example 2.21 implies that $\mathcal{O}_X(V)$ is an integral domain. Since $0 = (ab)|_V = a|_V b|_V$, either $a|_V = 0$ or $b|_V = 0$.

Hence for every $x \in U$ there exists an open (affine) neighborhood of $x$ where $a = 0$ or an open neighborhood, where $b = 0$. So either $x \notin W_a$ or $x \notin W_b$. In other words $W_a \cap W_b = \emptyset$. So by irreducibility, one of the sets is empty; say $W_a$. But then $a$ would lie in the intersection of all prime ideals, i.e. in the nilradical which is assumed to be just 0. Thus $a = 0$ and $\mathcal{O}_X(U)$ is integral as required. $\square$

**Definition 2.24.** Let $X$ be an integral scheme. By exercise A.20, the local ring $\mathcal{O}_{X,\eta}$ at the generic point $\eta$ of $X$ (i.e. $\eta$ is dense in $X$) is a field. It is called the *function field* of $X$ and denoted $K(X)$.

**Definition 2.25.** A scheme $X$ is called *Noetherian* if it can be covered by finitely many open affine subsets $U_i = \operatorname{Spec} A_i$ such that all $A_i$ are Noetherian rings.

**Definition 2.26.** Let $(X, \mathcal{O}_X)$ be a scheme. An *open subscheme* of $X$ is an open subset $U$ of $X$ with structure sheaf $\mathcal{O}_X|_U$. A morphism $f\colon Y \to X$ is called an *open immersion* if $f$ is an isomorphism of $Y$ with an open subscheme $U$ of $X$.

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

**Definition 2.27.** Let $(X, \mathcal{O}_X)$ be a scheme. A *closed immersion* is a morphism of schemes $f\colon Y \to X$ such that $f$ is a homeomorphism of $Y$ with a closed subset of $X$ and $\mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective. A *closed subscheme* of $X$ is an equivalence class of closed immersions, where we say that two closed immersions $f\colon Y \to X$ and $f'\colon Y' \to X$ are equivalent if there exist an isomorphism $g\colon Y \to Y'$ with $f = f' \circ g$.

*Example* 2.28. Let $X = \operatorname{Spec} A$ and take an ideal $\mathfrak{a} \lhd A$. Consider the scheme $Y = \operatorname{Spec}(A/\mathfrak{a})$. The canonical homomorphism $A \to A/\mathfrak{a}$ corresponds to a morphism $f\colon Y \to X$. Obviously $f(Y) = V(\mathfrak{a})$ and one can check that $f$ is a homeomorphism onto the closed set $V(\mathfrak{a})$.

For any point $x \in f(Y)$ (say $x = f(y)$) we have a homomorphism of local rings $A_x = (\mathcal{O}_X)_x \to (f_* \mathcal{O}_Y)_x = (\mathcal{O}_Y)_y = (A/\mathfrak{a})_{x/\mathfrak{a}}$ which is surjective. For $x \notin f(Y)$, the map $(\mathcal{O}_X)_x \to (f_* \mathcal{O}_Y)_x = 0$ is trivially surjective. Hence the morphism $\mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective, so $f$ is a closed immersion

Also note that for each $n \in \mathbb{N}$, $\mathfrak{a}^n$ defines a closed immersion $f_n\colon Y_n = \operatorname{Spec}(A/\mathfrak{a}^n) \to X$. But $f_n(Y_n) = V(\mathfrak{a}^n) = V(\mathfrak{a})$. Hence $V(\mathfrak{a}) \subseteq X$ can carry infinitely many different subscheme structures.

*Remark** 2.29. If $Y$ is any closed subset of $X$, then, as we just saw, in general there exist many different closed subscheme structures on $Y$. However there exits a unique reduced closed subscheme structure on $Y$. This structure is called the *induced reduced structure* on $Y$. See [Liu02, proposition 2.4.2c] or [Har77, example II.3.2.6].

**Definition 2.30.** Let $S$ be a scheme. A *scheme over $S$* or an *$S$-scheme* is a scheme $X$ together with a morphism $X \to S$. A morphism $X \to Y$ is a *morphism of schemes over $S$* if the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

*Remark** 2.31. If $S = \operatorname{Spec} A$ for some ring $A$, we say that a scheme over $\operatorname{Spec} A$ is a scheme over $A$ or an $A$-scheme. By exercise A.8, $\operatorname{Spec} \mathbb{Z}$ is the terminal object in the category of schemes, so every scheme can be seen as a scheme over $\mathbb{Z}$.

**Definition 2.32.** Let $S$ be a scheme and $X, Y$ two $S$-schemes. The *fibred product* of $X$ and $Y$ over $S$ is an $S$-scheme $X \times_S Y$ together with two morphism of $S$-schemes $p\colon X \times_S Y \to X$ and $q\colon X \times_S Y \to Y$ (called *projection maps*), such that the following universal property holds:

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

Whenever $Z$ is an $S$-scheme and $f\colon Z \to X$, $g\colon Z \to Y$ are two morphisms of $S$-schemes, there exists a unique morphism $\phi\colon Z \to X \times_S Y$ of $S$-schemes such that the following diagram commutes:

$$\begin{tikzcd} & & Y \arrow[dr] & \\ Z \arrow[r, dashed, "\phi"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "f"'] & X \times_S Y \arrow[ur, "q"] \arrow[dr, "p"'] & & S \\ & & X \arrow[ur] & \end{tikzcd}$$

**Theorem 2.33.** *The fibred product exists and is unique up to unique isomorphism.*

[Har77, theorem II.3.3], [Liu02, proposition 3.1.2]

*Proof.* Uniqueness is clear form the universal property.

Suppose $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} C$. We will show that $X \times_S Y = \operatorname{Spec}(B \otimes_A C)$. Let $Z$ be another scheme such that we have a commutative diagram (using theorem 2.12)

$$\begin{tikzcd} & & Y \arrow[dr] & \\ Z \arrow[urr, bend left, "g"] \arrow[drr, bend right, "f"'] & X \times_S Y \arrow[ur, "q"] \arrow[dr, "p"'] & & S \\ & & X \arrow[ur] & \end{tikzcd}$$

We need to show that there exists a unique morphism $Z \to X \times_S Y$ fitting into the diagram. By exercise A.8, this is the same as showing that there is a unique homomorphism of rings $B \otimes_A C \to \mathcal{O}_Z(Z)$ fitting into the diagram

$$\begin{tikzcd} & & B \arrow[dl, "q"'] \arrow[dll, bend right, "g"'] & \\ \mathcal{O}_Z(Z) & B \otimes_A C \arrow[l, dashed] & & A \arrow[ul] \arrow[dl] \\ & & C \arrow[ul, "p"] \arrow[ull, bend left, "f"] & \end{tikzcd}$$

But this is just the universal property of the tensor product.

For the general case, we need to cover $X$, $Y$ and $S$ by affine schemes and construct $X \times_S Y$ locally in the affine open sets and finally glue all of them together. See the references. $\square$

**Lemma* 2.34.** *Let $S$ be a scheme and let $X$ and $Y$ be $S$-schemes. Then*

[Liu02, proposition 3.1.4]

1. *For any $S$-scheme $Z$, there are canonical isomorphisms*

$$X \times_S S \cong X, \quad X \times_S Y \cong Y \times_S X, \quad (X \times_S Y) \times_S Z \cong X \times_S (Y \times_S Z).$$

2. *Let $Z$ be a $Y$-scheme and consider it as an $S$-scheme via $Z \to Y \to S$, then there exists a canonical isomorphism*

$$(X \times_S Y) \times_Y Z \cong X \times_S Z,$$

*where $X \times_S Y$ is a $Y$-scheme via the second projection map.*

3. *For morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ of $S$-schemes, there exists a unique morphism of $S$-schemes $f \times g\colon X \times_S Y \to X' \times_S Y'$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ {\scriptstyle p}\uparrow & & \uparrow{\scriptstyle p'} \\ X \times_S Y & \xrightarrow{f \times g} & X' \times_S Y' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ Y & \xrightarrow{\quad g \quad} & Y' \end{array}$$

   *commutes.*

4. *Let $i\colon U \to X$ and $j\colon V \to Y$ be open subschems. Then the momrphism $i \times j$ gives an isomorphism*
$$U \times_S V \cong p^{-1}(U) \cap q^{-1}(V) \subseteq X \times_S Y.$$

If $y$ is any point of a scheme $Y$, then the composition $\mathcal{O}_Y(Y) \to (\mathcal{O}_Y)_y \to k(y)$ induces a canonical morphism $\operatorname{Spec} k(y) \to Y$ (see exercise A.9).

**Definition 2.35.** Let $f\colon X \to Y$ be a morphism of schemes. The *fibre* of $f$ over $y \in Y$ is

$$X_y = X \times_Y \operatorname{Spec} k(y).$$

*Example* 2.36. Let $X$ be a scheme. There is a unique morphism $f\colon X \to \operatorname{Spec} \mathbb{Z}$ which corresponds to $\mathbb{Z} \to \mathcal{O}_X(X)$. Set $Y = \operatorname{Spec} \mathbb{Z}$. For the point $y = (0) \in Y$, the residue field is $k((0)) = \mathbb{Z}/\mathfrak{m}_0 = \mathbb{Q}$. Then $X_y = X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Q}$ (if $Y = \operatorname{Spec} A$, then $X_y = \operatorname{Spec}(A \otimes_{\mathbb{Z}} \mathbb{Q})$).

If $y = (p)$ for a prime number $p$, then $k(y) = \mathbb{Z}_p/\mathfrak{m} = \mathbb{F}_p$ (the finite field of $p$ elements). Thus $X_y = X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p$ (if $X = \operatorname{Spec} A$, then $X_y = \operatorname{Spec}(A \otimes_{\mathbb{Z}} \mathbb{F}_p)$). This is called *reduction mod p*.

*Example* 2.37. Let be $X$ a scheme over a field $K$ (i.e. over $\operatorname{Spec} K$) and let $K \subset L$ be a field extension. The fibred product $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is a scheme over $L$. In particular, $\mathbb{A}^n_K \times_{\operatorname{Spec} K} \operatorname{Spec} L \cong \mathbb{A}^n_L$, as $K[t_1, \dots, t_n] \otimes_K L \cong L[t_1, \dots, t_n]$.

*Example* 2.38. Let $f\colon X \to Y$ be a morphism of schemes with $Y$ integral. Then $Y$ has a unique generic point $\eta$. The *generic fibre* of $f$ is defined to be $X_\eta$. By definition $X_\eta$ is a scheme over $k(\eta) = K(Y)$, the function field of $Y$.

## 2.3. Proj and Projective Space

**Definition 2.39.** Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring. We set

$$\operatorname{Proj} S = \{\mathfrak{p} \in \operatorname{Spec} S : \mathfrak{p} \text{ is homogeneous and } \bigoplus_{d \geq 1} S_d \nsubseteq \mathfrak{p}\}.$$

For any homogeneous ideal $\mathfrak{a}$ of $S$ we define $V_+(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{a} \supseteq \mathfrak{p}\}$ and for any homogeneous $b \in S$, we set $D_+(b) = \operatorname{Proj} S \setminus V_+((b))$.

**Lemma 2.40.** *Let $S$ be a graded ring. Then:*

1. *For any homogeneous ideals $\mathfrak{a}, \mathfrak{b} \lhd S$: $V_+(\mathfrak{a}) \cup V_+(\mathfrak{b}) = V_+(\mathfrak{a} \cap \mathfrak{b}) = V_+(\mathfrak{a}\mathfrak{b})$.*

2. *For any family $\{\mathfrak{a}_i\}$ of homogeneous ideals of $S$: $\bigcap V_+(\mathfrak{a}_i) = V_+(\sum \mathfrak{a}_i)$.*

*Proof.* As in the affine case. $\square$

By taking the $V_+(\mathfrak{a})$ to be the closed sets, $\operatorname{Proj} S$ becomes a topological space.

**Definition 2.41.** Let $S = \bigoplus S_d$ be a graded ring, set $X = \operatorname{Proj} S$ and let $M = \bigoplus M_d$ be a graded $S$-module. For any open $U \subseteq X$, define $\widetilde{M}(U)$ to be the set of functions

$$s\colon U \to \coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$$

such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{(\mathfrak{p})}$ and such that $s$ is locally a quotient $\frac{m}{b}$ with $m \in M$, $b \in S$ homogeneous and of the same degree, i.e. for each $\mathfrak{p} \in U$ there exist a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and homogeneous elements $b \in S$, $m \in M$ of the same degree such that for all $\mathfrak{q} \in V$, $b \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{b}$ in $M_{(\mathfrak{q})}$.

*Remark* 2.42. We use the same notation $\widetilde{M}$ both in the affine case and in the case of a graded module over $\operatorname{Proj} S$. It should usually be clear from the context which construction is meant.

**Definition 2.43.** Let $S$ be a ring, $X = \operatorname{Proj} S$. We set $\mathcal{O}_X = \widetilde{S}$. So, for an open $U \subseteq X$, define $\mathcal{O}(U)$ to be the set of functions

$$s\colon U \to \coprod_{\mathfrak{p}\in U} S_{(\mathfrak{p})}$$

such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and such that $s$ is locally a quotient of homogeneous elements in $S$ of the same degree, i.e. for each $\mathfrak{p} \in U$ there exist a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and homogeneous elements $a, b \in S$ of the same degree such that for all $\mathfrak{q} \in V$, $b \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{b}$ in $S_{(\mathfrak{q})}$. Then $\mathcal{O}_X$ with the obvious restriction maps (which are ring homomorphisms) is a sheaf on $X$ making $(X, \mathcal{O}_X)$ into a ringed space.

**Theorem 2.44.** *Let $S$ be a graded ring, $X = \operatorname{Proj} S$ and $M$ a graded $S$-module. Then:*

1. *For any $\mathfrak{p} \in X$, $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$. In particular, $(\mathcal{O}_X)_{\mathfrak{p}} = S_{(\mathfrak{p})}$.*
2. *For any homogeneous $b \in \bigoplus_{d\geq 1} S_d$ there exists an isomorphism of locally ringed spaces*

$$(D_+(b),\ \mathcal{O}_X|_{D_+(b)}) \cong \operatorname{Spec} S_{(b)}$$

   *which takes $\widetilde{M}|_{D_+(b)}$ to $\widetilde{M_{(b)}}$.*
3. *In particular, $(X, \mathcal{O}_X)$ is a scheme.*

*Proof.* not given $\square$

**Definition 2.45.** Let $A$ be any ring. The scheme $\operatorname{Proj}(A[t_1, \ldots, t_n])$ is denoted $\mathbb{P}^n_A$ and called the *projective $n$-space* over $A$.

*Example* 2.46. If $k$ is an algebraically closed field, then $\mathbb{P}^n_k$ is essentially the same as the classical $n$-dimensional projective space over $k$.

**Definition 2.47.** Let $Y$ be a scheme. The *projective $n$-space over $Y$* is $\mathbb{P}^n_Y = Y \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$. A morphism $f\colon X \to Y$ is called *projective*, if there exists $n$ such that $f$ factors into a closed immersion $X \to \mathbb{P}^n_Y$ followed by the projection $\mathbb{P}^n_Y \to Y$. A *quasi-projective morphism* is a morphism that factors into an open immersion followed by a projective morphism.

*Remark** 2.48. If $Y = \operatorname{Spec} A$, then $\mathbb{P}^n_Y = \mathbb{P}^n_A$ (see exercise A.26).

## 2.4. Morphisms

*Remark* 2.49. The following properties are all defined for morphisms of schemes. However, if $X$ is a scheme over $S$, $X$ is said to have one of these properties if the morphisms $X \to S$ has it. If no base scheme $S$ is given, then $X$ is considered to be a scheme over $\mathbb{Z}$ (see remark 2.31).

**Definition 2.50.** A morphism of schemes $f\colon X \to Y$ is *of finite type* if there exists a covering of $Y$ be open affine subschemes $V_i = \operatorname{Spec} B_i$, such that for each $i$, $f^{-1}(V_i)$ can be covered by finitely many open affine subsschems $U_{ij} = \operatorname{Spec} A_{ij}$, where each $A_{ij}$ is a finitely generated $B_i$-algebra.

**Definition 2.51.** Let $f\colon X \to Y$ be a morphism of schemes. The *diagonal morphism* is the unique map $\Delta\colon X \to X \times_Y X$ whose composition with both projection maps is the identity.

$$\begin{tikzcd}
& & X \arrow[dr, "f"] & \\
X \arrow[r, dashed, "\Delta"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "\mathrm{id}"'] & X \times_Y X \arrow[ur] \arrow[dr] & & Y \\
& & X \arrow[ur, "f"'] &
\end{tikzcd}$$

The morphism $f$ is called *separated* if $\Delta$ is a closed immersion.

[Liu02, proposition 3.3.1] *Remark** 2.52. Separatedness is somewhat related to the Hausdorff property of topological spaces (note that schemes are in general not Hausdorff): A topological space $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in the product space $X \times X$. See also [EH00, exercise III-3].

*Example* 2.53. Let $K$ be a field. Then $\mathbb{A}^1_K \to \operatorname{Spec} K$ is a separated morphism.

**Theorem 2.54.** *Every morphism of of affine schemes is separated.*

*Proof.* Let $f\colon X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ be a morphism of affine schemes, given by a homomorphism $B \to A$. Then $X \times_Y X = \operatorname{Spec}(A \otimes_B A)$ and the diagonal morphism corresponds to the homomorphism $A \otimes_B A \to A$ given by $a \otimes a' \mapsto aa'$. This is surjective, so by example 2.28 the diagonal morphism $\Delta$ is a closed immersion. $\square$

**Theorem 2.55.** *An arbitrary morphism $f\colon X \to Y$ is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.*

*Proof.* The "only if"-direction is true by definition. So assume that $\Delta(X)$ is a closed subset of $X$. We have to show that $\Delta$ is actually a closed immersion. Consider the first projection $p\colon X \times_Y X \to X$. Then $p \circ \Delta = \mathrm{id}_X$; hence $\Delta$ is a homeomorphism onto its image, which is closed. It remains to be shown that $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective. This is a local question. For $Q \notin \Delta(X)$, the induced map on the stalk at $Q$ is certainly surjective. So let $P \in X$ and choose an open affine neighborhood $U$ of $P$ in $X$ such that $f(U)$ is contained in an open affine subset of $V$ of $Y$. Then $U \times_V U$ is an open affine neighborhood of $\Delta(P)$. By theorem 2.54 above, $\Delta\colon U \to U \times_V U$ is separated. Therefore the map of sheaves is surjective in a neighborhood of $\Delta(P)$. Since $P$ was arbitrary, $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective as required. $\square$

*Remark* 2.56. A local ring $R$ inside a field $L$ is a *valuation ring* if for any other local ring $A$ with $R \overset{\alpha}{\hookrightarrow} A \hookrightarrow L$ and $\alpha$ local, $R = A$.

**Theorem 2.57** (Valuative Criterion for Separatedness)**.** *Let $X$ be a Noetherian scheme. A morphism $f\colon Y \to X$ is separated if and only if the following holds:*

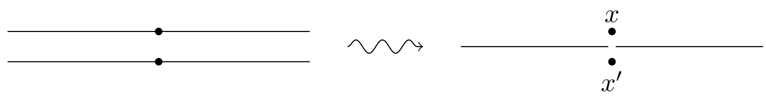
*For every valuation ring $R$ with fraction field $K$ and every commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "g"] & Y \end{tikzcd} \tag{2.1}$$

*where $\iota$ is induced by the inclusion $R \hookrightarrow K$, there exists at most one morphism $g\colon \operatorname{Spec} R \to X$ fitting into the diagram.*

*Proof.* Not given. (See [Har77, theorem II.4.3].) □

*Example* 2.58. Let $k$ be a field and $U = V = \mathbb{A}^1_k$. We glue $U$ and $V$ along the open subschemes $U \setminus \{(t)\}$, $V \setminus \{(t)\}$ and call the result $X$.




Let $x$, $x'$ be the two origins. Let $K$ be the fraction field of $\mathcal{O}_x \cong \mathcal{O}'_x$. By exercise A.10, there are morphisms $g$, $g'$ making the following diagram commutative:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] & \operatorname{Spec} K \arrow[l] \arrow[d] \arrow[ll, bend right, "\cong"'] \\ \operatorname{Spec} \mathcal{O}_x \arrow[r] \arrow[ur, "g"] & \operatorname{Spec} k & \operatorname{Spec} \mathcal{O}_{x'} \arrow[l] \arrow[ul, "g'"'] \arrow[ll, bend left, "\cong"] \end{tikzcd}$$

If we set $R = \mathcal{O}_x \cong \mathcal{O}_{x'}$, we have two different morphism from $\operatorname{Spec} R$ to $X$ making (2.1) commute. Hence $X$ is not separated over $\operatorname{Spec} k$.

**Lemma 2.59.** *Closed immersions of Noetherian schemes are separated.*

This is also true for non-Noetherian schemes, see [Liu02, proposition 3.3.9]

*Proof.* Let $f\colon X \to Y$ be a closed immersion. Suppose that we have a commutative diagram like (2.1). We have to show that there is at most one $g$ that fits into it.

Let $y$ be the image of $\mathfrak{m}_R$ in $Y$ and $y'$ the image of the generic point of $\operatorname{Spec} R$. Since the diagram is commutative, $y' \in f(X)$. As $f(X)$ is closed, it must also contain $y$. Take any open affine subset $U$ of $Y$ with $y \in U$ and set $V = f^{-1}U$. This gives a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & V \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "h"] & U \arrow[r, hook] & Y \end{tikzcd}$$

Since $h^{-1}U$ is open, it contains the generic point and thus $y' \in U$. Hence it is enough to look at the left side of the diagram. Let $U = \operatorname{Spec} A$. Since $V \to U$ is a closed immersion, $V = \operatorname{Spec} A/\mathfrak{a}$ for some ideal $\mathfrak{a} \lhd A$ (see exercise A.21 or corollary 2.87). By theorem 2.12 we get a commutative diagram on the level of rings:

$$\begin{tikzcd} K & A/\mathfrak{a} \arrow[l] \\ R \arrow[u, hook] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Since $A \to A/\mathfrak{a}$ is surjective, there is at most one homomorphism $A/\mathfrak{a} \to R$ that fits into the diagram. $\square$

**Definition 2.60.** A morphism of schemes $f\colon X \to Y$ is *universally closed* if for every morphism $S \to Y$ and every closed subset $T$ of $S \times_Y X$, the image of $T$ under the projection map is closed in $S$.

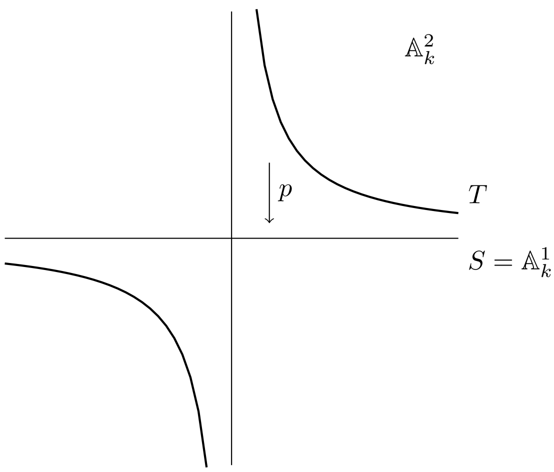
$$\begin{tikzcd} T \arrow[r, hook] \arrow[d] & S \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\ p(T) \arrow[r, hook] & S \arrow[r] & Y \end{tikzcd}$$

*Remark** 2.61. By setting $S = Y$ and using lemma 2.34 we see that if $f$ is universally closed, then in particular $f$ is closed.

**Definition 2.62.** A morphism is called *proper* if it is separated, of finite type and universally closed.

*Remark** 2.63. In general topology a continuous function $f\colon X \to Y$ is called proper if the preimage of every compact set is compact. If $X$ is Hausdorff and $Y$ locally compact, then $f$ is proper if and only if for any space $Z$, the map $Z \times X \to Z \times Y$ given by $(z, x) \mapsto (z, f(x))$ is closed.

*Example* 2.64. Let $X = \mathbb{A}^1_k$ for a field $k$ and consider $f\colon X \to \operatorname{Spec} k$. We will show that $f$ is not universally closed and hence is not proper. Take $S = \mathbb{A}^1_k$ and $T \subseteq \mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k = \mathbb{A}^2_k$ to be $V((t_1 t_2 - 1))$. The image of $T$ under the projection $p$ is $\mathbb{A}^1_k \setminus \{(t)\}$, which is not closed.




**Theorem 2.65** (Valuative Criterion for Properness)**.** *Let $X$ be Noetherian and $f\colon X \to Y$ a morphism of finite type. Then $f$ is proper if and only if the following holds:*

*For every valuation ring $R$ with fraction field $K$ and every commutative diagram*

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "g"] & Y
\end{tikzcd} \tag{2.2}$$

*where $\iota$ is induced by the inclusion $R \hookrightarrow K$, there exists exactly one morphism $g\colon \operatorname{Spec} R \to X$ fitting into the diagram.*

*Proof (Sketch).* First suppose that $f$ is proper and that we are given a diagram as in the statement. Since $f$ is separated, there exists at most one $g$ fitting into it (theorem 2.57). So we only have to show the existence of $g$. Set $U = \operatorname{Spec} K$ and $Z = \operatorname{Spec} R$. Consider the fibred product

*see also* [Har77, theorem 4.7]

$$\begin{tikzcd}
U \arrow[dr, "h"] \arrow[drr] \arrow[ddr] & & \\
& Z \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\
& Z \arrow[r] & Y
\end{tikzcd}$$

where $h$ exists by the universal property. Put $a = h(U)$, $T = \overline{\{a\}} \subseteq Z \times_Y X$. The proper map $f$ is universally closed, so $p(T)$ is closed in $Z$. Thus $p(T)$ contains the maximal ideal $\mathfrak{m}_R$. Choose $b \in T$ such that $p(b) = \mathfrak{m}_R$. Take the reduced induced subscheme structure on $T$. Obviously $T$ is irreducible and hence integral. There is a local homomorphism $R = (\mathcal{O}_Z)_{\mathfrak{m}_R} \to \mathcal{O}_{T,b}$. Also, $a$ is the generic point of $T$ and by construction we have a homomorphism $k(a) \hookrightarrow K$. By integrality, $\mathcal{O}_{T,b} \hookrightarrow k(a)$. This situation induces a morphism $Z \to T$ and so a morphism $g\colon Z \to X$ which fits into the diagram (without proof).

Now suppose that for every diagram like (2.2) we have a unique morphism $g$. We have to prove that $f$ is proper. By assumption, $f$ is of finite type and by theorem 2.57 it is separated. So we only need to verify that $f$ is universally closed. Hence suppose we are given a morphism $S \to Y$ and suppose that $T \subseteq S \times_Y X$ is closed. We put the reduced induced structure on $T$.

$$\begin{tikzcd}
T \arrow[r, hook] \arrow[dr] & S \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\
& S \arrow[r] & Y
\end{tikzcd}$$

It turns out that in order to prove that $p(T)$ is closed, it is enough to show that $\overline{\{a\}} \subseteq p(T)$ for all $a \in p(T)$. So pick any $a \in p(T)$ with $p(t) = a$ for some $t \in T$. Set $S' = \overline{\{a\}}$ and let $b \in S'$. We have to show that $b \in p(T)$. Put the reduced induced structure on $S'$ (it is integral as above). By exercise A.10, there exists a morphism $\operatorname{Spec} \mathcal{O}_{S',b} \to S'$. Also, $a$ is the generic point of $S'$, so that there are inclusions $\mathcal{O}_{S',b} \hookrightarrow k(a) \hookrightarrow k(t)$. By commutative algebra [?], there exists a valuation ring $R$ for $k(t)$ such that $\mathcal{O}_{S',b} \to R$ is a local homomorphism. This gives a diagram

$$\begin{tikzcd}
\operatorname{Spec} k(t) \arrow[rr] \arrow[d] & & T \arrow[r] & S \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[urrr, dashed, "h"] \arrow[urrrr, dashed, "g"'] & \operatorname{Spec} \mathcal{O}_{S',b} \arrow[r] & S' \arrow[r] & S \arrow[r] & Y
\end{tikzcd}$$

By the hypothesis, there exists a unique morphism $g\colon \operatorname{Spec} R \to X$ fitting into the diagram. By the universal property of the fibred product $S \times_Y X$, $g$ determines a unique morphism $h\colon \operatorname{Spec} R \to S \times_Y X$. The morphism $h$ has to send the generic point of $\operatorname{Spec} R$ into $T$. Since $T$ is closed that implies $h(\operatorname{Spec} R) \subseteq T$. In particular, there exists $t' \in T$ such that $h(\mathfrak{m}_R) = t'$. The commutativity of the diagram ensures that $p(t') = b$, so that $b \in p(T)$ as required. $\square$

This is also true for non-Noetherian schemes, see [Liu02, proposition 3.3.16]

**Lemma 2.66.** *For Noetherian schemes the following statements hold:*

1. *Closed immersions are proper.*
2. *Compositions of proper morphisms are proper.*

*Proof.*

1. Proceed as in the proof of lemma 2.59. Note that there must always exist a homomorphism $A/\mathfrak{a} \to R$ that fits into the last diagram. Hence we only have to show that $V \to U$ is of finite type. But $A/\mathfrak{a}$ is obviously a finitely generated $A$-algebra.

2. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be proper morphisms. The composition $g \circ f$ is of finite type, so we can use the valuative criterion. Suppose we have a commutative diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \arrow[dd, bend left=60, "gf"] \\ \operatorname{Spec} K \arrow[ur] \arrow[r, dashed, "e"] \arrow[d] & Y \arrow[d, "g"] \\ \operatorname{Spec} R \arrow[uur, dashed] \arrow[ur, dashed, "h"'] \arrow[r] & Z \end{tikzcd}$$

   Since $g$ is proper there exists a unique morphism $h$ and since $f$ is proper we get a unique morphism $e$. So $g \circ f$ is proper. $\square$

[Har77, thm II.4.9]

Again, this is also true for non-Noetherian schemes, see [Liu02, proposition 3.3.30] or [GD61a, thm. 5.5.3].

**Theorem 2.67.** *Every projective morphism of Noetherian schemes is proper.*

*Proof.* Let $f\colon X \to Y$ be a projective morphism of Noetherian schemes. Since $f$ is the composition of a closed immersion and the projection $\mathbb{P}^n_Y \to Y$, it is of finite type. So we can apply the valuative criterion for properness. Suppose we have a commutative diagram ($R$ a valuation ring of $K$)

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[r, "g"] \arrow[d, "f"] & \mathbb{P}^n_Y \arrow[r] \arrow[dl, "p"] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d, "q"] \\ \operatorname{Spec} R \arrow[r] & Y \arrow[rr] & & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

As $g$ is a closed immersion, it is proper by lemma 2.66. Hence it is enough to prove that $p$ is proper. By the universal property of fibred products, we see that there exists a unique map $\operatorname{Spec} R \to \mathbb{P}^n_Y$ fitting into the diagram if and only if there exists a unique fitting map $\operatorname{Spec} R \to \mathbb{P}^n_{\mathbb{Z}}$. So it suffices to show that $q\colon \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper. We have reduced the situation to the following commutative diagram:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "e"] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d, "q"] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

The projective $n$-space $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}\mathbb{Z}[t_0, \dots, t_n]$ is covered by the open subsets $U_i = D_+(t_i)$ for $i = 0, \dots, n$ and by theorem 2.44 these are

$$U_i = D_+(t_i) = \operatorname{Spec}\mathbb{Z}[t_0, \dots, t_n]_{(t_i)} = \operatorname{Spec}\mathbb{Z}\left[\frac{t_0}{t_i}, \dots, \frac{t_n}{t_i}\right].$$

Let $a = e(\operatorname{Spec} K)$. If $a \notin U_i$ for some $i$, then $a \in V_+(t_i) = \mathbb{P}^{n-1}_{\mathbb{Z}}$, so that we can do induction. (The base case $n = 0$ is trivial.)

Therefore we can assume that $a \in \bigcap_i U_i$. In that case each $\frac{t_j}{t_i}$ is an element of $\mathcal{O}_a$. Further, since $\frac{t_i}{t_j}$ is also in $\mathcal{O}_a$, these elements are all invertible in $\mathcal{O}_a$. The morphism $\operatorname{Spec} K \to \mathbb{P}^n_{\mathbb{Z}}$ gives an inclusion $k(a) \to K$. Let $s_{ji}$ be the image of $\frac{t_j}{t_i}$ under this inclusion (and the projection $\mathcal{O}_a \to k(a)$).

By commutative algebra [?], there exists a valuation $v\colon K \to G$ (where $G$ is a totally ordered Abelian group) such that

$$\begin{aligned} v(\mu\vartheta) &= v(\mu) + v(\vartheta) & R &= \{\mu \in K : v(\mu) \geq 0\} \\ v(\mu + \vartheta) &\geq \min\{v(\mu), v(\vartheta)\} & \mathfrak{m}_R &= \{\mu \in K : v(\mu) > 0\} \end{aligned}$$

Note that $v(s_{jl}) = v(s_{ji}) + v(s_{il})$. Fix an index $j$ such that $v(s_{j0})$ is minimal among $\{s_{10}, \dots, s_{n0}\}$. So, for each $i$ we have $v(s_{ij}) = v(s_{i0}) - v(s_{j0}) \geq 0$, i.e. $s_{ij} \in R$ for all $i$. This gives a natural homomorphism $\mathbb{Z}\left[\frac{t_0}{t_j}, \dots, \dots, \frac{t_n}{t_j}\right] \to R$ which in turn gives a morphism $\operatorname{Spec} R \to U_j \subseteq \mathbb{P}^n_{\mathbb{Z}}$ that fits into the diagram.

It remains to be shown that there is at most one such morphism. If there was another such morphism, then the image had to lie in another $U_i$. But then both $s_{ji}$ and $s_{ij}$ would lie in $R^*$ which is not possible. □

**Definition 2.68.** A morphism $f\colon X \to Y$ is *quasi-compact* if there exists a cover of $Y$ by open affines $Y_i$ such that $f^{-1}(Y_i)$ is quasi-compact for each $i$.

*Remark** 2.69. $f$ is quasi-compact if and only if for every cover of $Y$ by open affines $Y_i$ the preimages $f^{-1}(Y_i)$ are quasi-compact for all $i$. [Har77, Exercise II.3.2]

## 2.5. Sheaves of Modules

**Theorem 2.70.** *Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine schemes. Let $M$, $M_i$ be $A$-modules, $N$ a $B$-module and $f\colon X \to Y$ a morphism of schemes. Then:* [Har77, Prop. II.5.2]

1. $\widetilde{M_1 \otimes_A M_2} = \widetilde{M_1} \otimes_{\mathcal{O}_X} \widetilde{M_2}$
2. $\widetilde{\bigoplus M_i} = \bigoplus \widetilde{M_i}$
3. $f_*\widetilde{M} = \widetilde{{}_BM}$, *where ${}_BM$ is $M$ considered as a $B$-module.*
4. $f^*\widetilde{N} = \widetilde{N \otimes_B A}$

*Proof.*

1. Let $\mathcal{P}$ be the presheaf given by $\mathcal{P}(U) = \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U)$. We will define a morphism $\phi\colon \mathcal{P} \to \widetilde{M \otimes_A N}$. Let $U$ be an open subset of $X$, $s \in \widetilde{M}(U)$, $t \in \widetilde{N}(U)$. Define an element $r \in \widetilde{M \otimes_A N}(U)$ as

$$r\colon U \to \coprod_{\mathfrak{p} \in U} (M \otimes_A N)_{\mathfrak{p}} \cong \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}, \quad r(\mathfrak{p}) = s(\mathfrak{p}) \otimes t(\mathfrak{p}).$$

If $s$ is locally given by $\frac{m}{a}$ and $t$ by $\frac{n}{b}$, then $r$ is locally given by $\frac{m\otimes n}{ab}$. The association $(s,t) \mapsto r$ is bilinear and hence descends to a homomorphism $\phi_U : \mathcal{P}(U) \to \widetilde{M \otimes_A N}(U)$. This gives the morphism (of presheaves) $\phi\colon \mathcal{P} \to \widetilde{M \otimes_A N}$, which in turn gives a morphism (of sheaves)

$$\phi^+ : \mathcal{P}^+ = \widetilde{M} \otimes_{O_X} \widetilde{N} \to \widetilde{M \otimes_A N}.$$

For any $\mathfrak{p} \in X$ the homomorphism

$$\phi^+_{\mathfrak{p}} = \phi_{\mathfrak{p}} : \mathcal{P}^+_{\mathfrak{p}} = \mathcal{P}_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \longrightarrow \widetilde{M \otimes_A N}_{\mathfrak{p}} \cong (M \otimes_A N)_{\mathfrak{p}}$$

is an isomorphism (see also exercise A.28). Hence $\phi^+$ is an isomorphism.

2. The proof is like the one of the first statement, only simpler.

3. We will define a morphism $\psi\colon \widetilde{{}_BM} \to f_*\widetilde{M}$. Let $U \subseteq Y$ be open. Pick an $s \in \widetilde{{}_BM}(U)$. Then $s$ is by definition a function $U \to \coprod_{\mathfrak{q}\in U} ({}_BM)_{\mathfrak{q}}$. If $f(\mathfrak{p}) = \mathfrak{q}$, then we have a homomorphism $\beta_{\mathfrak{p}} : ({}_BM)_{\mathfrak{q}} \to M_{\mathfrak{p}}$ (as we have a ring homomorphism $B \to A$). Define $\psi_U(s)$ to be the function

$$f^{-1}U \to \coprod_{\mathfrak{p}\in f^{-1}U} M_{\mathfrak{p}}, \quad \mathfrak{p} \mapsto \beta_{\mathfrak{p}}\left(s\left(f\left(\mathfrak{p}\right)\right)\right) \in M_{\mathfrak{p}}.$$

Let $\alpha\colon B \to A$ be the ring homomorphism corresponding to $f\colon X \to Y$. If $s$ is locally given by $\frac{m}{b}$ ($m \in {}_BM$, $b \in B$), then $\psi_U(s)$ is locally given by $\frac{m}{\alpha(b)}$. Observe that if $b \in B$, then $f^{-1}D(b) = D(\alpha(b))$. Now

$${}_BM_b \cong \widetilde{{}_BM}(D(b)) \xrightarrow{\psi_{D(b)}} (f_*\widetilde{M})(D(b)) = \widetilde{M}(f^{-1}D(b)) = \widetilde{M}(D(\alpha(b))) \cong M_{\alpha(b)}$$

is the natural $B$-module isomorphism between the two outer modules. Thus $\psi$ is an isomorphism.

4. Proof ommited. □

*Remark* 2.71. For $A$-modules $M$ and $N$ and $X = \operatorname{Spec} A$, there exists a natural isomorphism $\operatorname{Hom}_A(M,N) \cong \operatorname{Hom}_{O_X}(\widetilde{M}, \widetilde{N})$ (see exercise A.29).

*Remark* 2.72. Let $X$ be $\operatorname{Spec} A$. Any sequence of $A$-modules

$$\cdots \to M_{-1} \to M_0 \to M_1 \to \dots \tag{2.3}$$

induces a sequence of $O_X$-modules

$$\cdots \to \widetilde{M_{-1}} \to \widetilde{M_0} \to \widetilde{M_1} \to \dots . \tag{2.4}$$

The sequence (2.4) is exact if and only if (2.3) is exact. Indeed, the latter is exact if and only if all localized sequences

$$\cdots \to (M_{-1})_{\mathfrak{p}} \to (M_0)_{\mathfrak{p}} \to (M_1)_{\mathfrak{p}} \to \dots$$

are exact. These are just the sequences on the stalks induced by 2.4.

**Definition 2.73.** Let $X$ be a scheme. An $O_X$-module $\mathcal{F}$ is *quasi-coherent* if $X$ can be covered by open affine schemes $U_i = \operatorname{Spec} A_i$ such that $\mathcal{F}|_{U_i} = \widetilde{M_i}$ for some $A_i$-modules $M_i$. It is *coherent* if additionally all $M_i$ are finitely generated $A_i$-modules.

*Example* 2.74. For any scheme $X$, the structure sheaf $O_X$ is coherent. Indeed for any open affine subscheme $U = \operatorname{Spec} A$ we have $O_X|_U = \widetilde{A}$.

*Example* 2.75. Let $A$ be a DVR with maximal ideal $\mathfrak{m}$ and set $X = \operatorname{Spec} A$. This scheme has only three open sets: $\emptyset$, $X$ and $U = X \setminus \{\mathfrak{m}\} = \{\eta\}$, where $\eta$ is the generic point of $X$ (corresponding to the zero ideal). Let $K = k(\eta)$. Let $\mathcal{F}$ be the sheaf defined by $\mathcal{F}(X) = 0$, $\mathcal{F}(U) = K$. For any open affine covering $X = \bigcup U_i$, there must be one $U_i$ with $U_i = X$, otherwise the maximal ideal would not be contained in any of the open sets. Since $\widetilde{M}(X) = M$ there is no $A$-module $M$ such that $\widetilde{M} = \mathcal{F}$. So $\mathcal{F}$ is not quasi-coherent.

**Lemma 2.76.** *If $X$ is an affine scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module, then there exists a natural morphism $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$.*

*Proof.* To ease notation set $M = \mathcal{F}(X)$. To define a morphism of sheaves $\widetilde{M} \to \mathcal{F}$, it suffices to define compatible maps $\widetilde{M}(D(b)) \to \mathcal{F}(D(b))$ for each $b \in A$. By theorem 2.9, $\widetilde{M}(D(b)) \cong M_b$. Thus we can define the map $\widetilde{M}(D(b)) \to \mathcal{F}(D(b))$ by sending $\frac{m}{b^r}$ to $\frac{1}{b^r} \cdot m|_{D(b)}$ (this is defined since $\frac{1}{b^r}$ is an element of $\mathcal{O}_X(D(b))$). It is easy to check that these maps are well-defined and compatible. $\square$

**Corollary 2.77.** *Let $X$ be* $\operatorname{Spec} A$ *and let $M$ be an $A$-module. Then $\widetilde{M}|_{D(b)} \cong \widetilde{M_b}$.*

*Proof.* By theorem 2.9 and the preceding lemma, there is a morphism $\phi\colon \widetilde{M_b} \to \widetilde{M}|_{D(b)}$. For any point $\mathfrak{p} \in D(b)$, $\phi_p\colon (\widetilde{M_b})_{\mathfrak{p}} \cong M_{\mathfrak{p}} \to M_{\mathfrak{p}}$ is an isomorphism. Hence $\phi$ is an isomorphism. $\square$

**Theorem 2.78.** *Let $X$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for every open cover of $X$ by open affine schemes $U_i = \operatorname{Spec} A_i$, there are $A_i$-modules $M_i$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$.*

*Remark* 2.79. If $X$ is Noetherian, a similar statement holds for coherent sheaves with $M_i$ finitely generated.

*Proof.* One direction is obvious. So suppose $\mathcal{F}$ is quasi-coherent and choose any open affine subset $U = \operatorname{Spec} A$ of $X$. We have to find an $A$-module $M$ such that $\mathcal{F}|_U \cong \widetilde{M}$. By definition, for each point $p \in U$ there exists an open affine neighborhood $W = \operatorname{Spec} B$ of $p$ such that $\mathcal{F}|_W \cong \widetilde{N}$ for some $B$-module $N$. By the corollary we can replace $W$ by any principal open subset of itself. In particular, we can assume that $W \subseteq U$ and $W = D(b)$ for some $b \in A$. Hence we can reduce to the situation $X = U$ is affine, where we have to show that $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$. Using the corollary again, we see that in this case $X$ is covered by finitely many $D(b_i)$ such that for all $i$ we have $\mathcal{F}|_{D(b_i)} \cong \widetilde{M_i}$ for an $A_{b_i}$-module $M_i$.

Let $f_i\colon D(b_i) \to X$ and $f_{ij}\colon D(b_ib_j) \to X$ be the inclusion maps. Note that $D(b_i) \cap D(b_j) = D(b_ib_j)$. Set

$$\mathcal{G} = \bigoplus f_{i*}\left(\mathcal{F}|_{D(b_i)}\right) \qquad \text{and} \qquad \mathcal{L} = \bigoplus f_{ij*}\left(\mathcal{F}|_{D(b_ib_j)}\right).$$

Define a map $\phi\colon \mathcal{F} \to \mathcal{G}$ by sending $s \in \mathcal{F}(V)$ to $(s_1, \ldots, s_n)$ with $s_i = s|_{D(b_i)\cap V}$. Further define $\psi\colon \mathcal{G} \to \mathcal{L}$ by sending $(s_i)_i \in \mathcal{G}(V)$ to $(s_i|_{D(b_ib_j)\cap V} - s_j|_{D(b_ib_j)\cap V})_{i,j}$. By lemma 1.4 (i.e. the sheaf condition) the sequence

$$0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{L} \tag{2.5}$$

is exact. By theorem 2.70.3 we have $f_{i*}\left(\mathcal{F}|_{D(b_i)}\right) = \widetilde{{}_AM_i}$ and $f_{ij*}\left(\mathcal{F}|_{D(b_ib_j)}\right) = \widetilde{{}_AM_{ij}}$ for some $A_{b_ib_j}$-modules $M_{ij}$. Taking global sections in (2.5) and using the sheaf condition (lemma 1.4) we obtain the exact sequence

$$0 \to \mathcal{F}(X) \to \bigoplus {}_AM_i \to \bigoplus {}_AM_{ij}.$$

Hence (by remark 2.72 and theorem 2.70.2) the sequence

$$0 \to \widetilde{\mathcal{F}(X)} \to \bigoplus \widetilde{{}_AM_i} \to \bigoplus \widetilde{{}_AM_{ij}} \tag{2.6}$$

is exact. From the sequences (2.5) and (2.6) we deduce that both $\mathcal{F}$ and $\widetilde{\mathcal{F}(X)}$ are isomorphic to $\ker\psi$. Thus $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$ as required. □

**Theorem 2.80.** *Let $X$ be a scheme and $\phi\colon \mathcal{F} \to \mathcal{G}$ a morphism of quasi-coherent sheaves on $X$. Then $\ker\phi$ and $\operatorname{im}\phi$ are also quasi-coherent. If $X$ as Noetherian and $\mathcal{F}, \mathcal{G}$ are coherent, then $\ker\phi$ and $\operatorname{im}\phi$ are coherent.*

*Proof.* Suppose $U = \operatorname{Spec} A$ is an open affine subset of $X$. Then there exist $A$-modules $M$ and $N$ such that $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \widetilde{N}$. So $\phi_U$ gives an $A$-module homomorphism $M \to N$. Let $L$ be the kernel of this homomorphism. This gives an exact sequence

$$0 \to \widetilde{L} \to \widetilde{M} \to \widetilde{N}.$$

Thus on $U$ we have $\ker\phi \cong \widetilde{L}$. Letting $U$ range over an open affine cover we see that $\ker\phi$ is quasi-coherent. The argument for $\operatorname{im}\phi$ is similar, but involves an additional step; see Exercise A.30.

For the case that $X$ is Noetherian, suppose that $M$ and $N$ are finitely generated and $A$ is Noetherian. Then $L$ is a submodule of a Noetherian module and hence finitely generated. Hence $\ker\phi$ is coherent and a similar argument shows that $\operatorname{im}\phi$ is coherent. □

**Theorem 2.81.** *Let $f\colon X \to Y$ be a morphism of schemes.*

1. *If $\mathcal{G}$ is a quasi-coherent (resp. coherent) sheaf on $Y$, then $f^*\mathcal{G}$ is quasi-coherent (resp. coherent) on $X$.*
2. *If $X$ is Noetherian (or if $f$ is separated and quasi-compact) and $\mathcal{F}$ is a quasi-coherent sheaf on $X$, then $f_*\mathcal{F}$ is quasi-coherent.*

*see also* [GD60, thm. 9.2.1]

*Proof.*

1. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Since the problem is local, we may assume that $Y$ is affine, say $Y = \operatorname{Spec} A$. Then there exists an $A$-module $M$ such that $\mathcal{G} = \widetilde{M}$. Let $U = \operatorname{Spec} B$ be an open affine subset of $X$. By theorem 2.70, $(f^*\mathcal{G})|_U \cong \widetilde{M \otimes_A B}$. So $f^*\mathcal{G}$ is quasi-coherent.

   Now assume that $\mathcal{G}$ is coherent. Then $M$ is finitely generated over $A$ and therefore $M \otimes_A B$ is finitely generated over $B$. So $f^*\mathcal{G}$ is indeed coherent.

[Har77, Prop. II.5.8c]

2. The problem is again local (in $Y$), so that we may assume that $Y = \operatorname{Spec} A$ is affine. Under either hypothesis we can cover $X$ with finitely many open affine subsets $U_i$. If $f$ is separated, then $U_i \cap U_j$ is affine. If $X$ is Noetherian, $U_i \cap U_j$ can be covered with finitely many open affine subsets $U_{ijk}$. (Include the separated case in this notation.) By the sheaf condition, we have an exact sequence
$$0 \to f_*\mathcal{F} \to \bigoplus_i f_*(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}}).$$
   The last two sheaves in this sequence are quasi-coherent by theorem 2.70. Thus $f_*\mathcal{F}$ is quasi-coherent by theorem 2.80. □

*Remark** 2.82. If $\mathcal{F}$ is coherent, then $f_*\mathcal{F}$ need not be coherent: Consider for example the inclusion $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec} Z$ and the sheaf $\widetilde{\mathbb{Q}}$. However, if $f$ is proper, then $f_*\mathcal{F}$ is coherent if $\mathcal{F}$ is coherent [GD61b, Théorème 3.2.1].

**Definition 2.83.** Let $X$ be a scheme. An *ideal sheaf* on $X$ is an $\mathcal{O}_X$-module $I \subseteq \mathcal{O}_X$ If $f\colon Y \to X$ is a closed immersion, then the kernel of $f^\#\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is called the *ideal sheaf of* $Y$ and denoted $I_Y$.

*Remark* 2.84. Recall that a closed subscheme is given by an equivalence class of closed immersions. The ideal sheaves of all the closed immersions in a class are the same. So really the ideal sheaf is an object associated with a closed subscheme.

**Theorem 2.85.** *Let $X$ be a scheme. Then there exists a natural bijection*

$$\{\textit{closed subschemes of } X\} \longleftrightarrow \{\textit{quasi-coherent ideal sheaves}\}.$$

*Remark* 2.86. If $X$ is Noetherian, all these quasi-coherent ideal sheaves are in fact coherent.

*Proof.* To each closed subscheme $Y$ of $X$ we associate the ideal sheaf $\mathcal{I}_Y$: The closed immersion $f: Y \to X$ is obviously quasi-compact and it is separated by lemma 2.59. Thus $f_*\mathcal{O}_Y$ is quasi-coherent (theorem 2.81) as is the kernel of $\mathcal{O}_X \to f_*\mathcal{O}_Y$ (theorem 2.80), i.e. $\mathcal{I}_Y$ is a quasi-coherent ideal sheaf on $Y$.

Conversely, suppose that $\mathcal{I}$ is a quasi-coherent ideal sheaf on $X$. Consider the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$$

Let $\Gamma$ be the set of points $p$ of $X$ with $(\mathcal{O}_X/\mathcal{I})_p = 0$. We will show that $\Gamma$ is an open subset of $X$. Let $U = \operatorname{Spec} A$ be an open affine subset of $X$. Then (by theorem 2.78) $\mathcal{I}|_U = \widetilde{\mathfrak{a}}$ for some ideal $\mathfrak{a}$ of $A$. Also, $(\mathcal{O}_X/\mathcal{I})|_U = \widetilde{A/\mathfrak{a}}$. A prime ideal $\mathfrak{p}$ of $A$ gives a point in $U \cap \Gamma$ if and only if its image in $A/\mathfrak{a}$ is 0. Thus $U \cap \Gamma = U \setminus V(\mathfrak{a})$ which is open. Letting $U$ range over an open affine cover, we see that $\Gamma$ is open. Hence $Y = X \setminus \Gamma$ is closed.

Let $f: Y \to X$ be the inclusion and consider $Y$ with the structure sheaf $f^{-1}(\mathcal{O}_X/\mathcal{I})$. Then $f_*\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$ (on any affine $U$ we have $f_*\mathcal{O}_Y = \widetilde{A/\mathfrak{a}}$) and if we let $f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Y$ be the projection, then $\mathcal{I}_Y = \ker f^{\#} = \mathcal{I}$. ☐

**Corollary 2.87.** *If $X = \operatorname{Spec} A$, then the closed subschemes of $X$ are in one-to-one correspondence with the ideals of $A$.*

*Proof.* In this case there is a one-to-one correspondence between quasi-coherent ideal sheaves $\mathcal{I}$ on $X$ and ideals $\mathfrak{a}$ of $A$, which is given by $\mathcal{I} = \widetilde{\mathfrak{a}}$. Thus the statement is a direct consequence of 2.85. ☐

**Theorem 2.88.** *If $M$ is a graded module over a graded ring $S$, then $\widetilde{M}$ is a quasi-coherent sheaf over $\operatorname{Proj} S$. If $S$ is Noetherian and $M$ finitely generated over $S$, then $\widetilde{M}$ is coherent.*

*Proof.* Not given. ☐

**Definition 2.89.** Let $S$ be a graded ring and $X = \operatorname{Proj} S$. Then the $\mathcal{O}_X$-module $\widetilde{S(n)}$ is denoted by $\mathcal{O}_X(n)$ (so that $\mathcal{O}_X(0) = \mathcal{O}_X$). For any $\mathcal{O}_X$-module $\mathcal{F}$ we put $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

*Remark** 2.90. $\mathcal{O}_X(1)$ is called the *twisting sheaf of Serre* and $\mathcal{F}(n)$ is the *twisted sheaf*.

**Theorem 2.91.** *Suppose that $S = \bigoplus_{d \geq 0} S_d$ is a graded ring that is generated by $S_1$ as an algebra over $S_0$. Let $X = \operatorname{Proj} S$ and let $M, N$ be graded $S$-modules. Then*

1. $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_S N}$

2. $\widetilde{M}(n) = \widetilde{M(n)}$. *In particular,* $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n)$.

3. *Let $\alpha: S \to T$ be a surjective graded homomorphism of graded rings and let $f: \operatorname{Proj} T \to \operatorname{Proj} S$ be the corresponding morphism (see exercises A.14 and A.22). Then $f^*\widetilde{M} \cong \widetilde{M \otimes_S T}$. If $K$ is a $T$-module, then $f_*\widetilde{K} \cong \widetilde{{}_SK}$.*

*Remark* 2.92. The condition "generated by $S_1$ as an algebra over $S_0$" is fulfilled by polynomial rings and their quotients.

*Proof.*

1. We will define a morphism $\phi\colon \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_S N}$. Since the open sets of the form $D_+(b)$ for homogeneous $b \in \bigoplus_{d \geq 1} S_d$ form a base of the topology of $X$, it suffices to define $\phi$ on these sets.

   Pick a homogeneous $b \in \bigoplus_{d \geq 1} S_d$. By theorems 2.44 and 2.70 we have

   $$\widetilde{M}|_{D_+(b)} \cong \widetilde{M_{(b)}}, \qquad \widetilde{N}|_{D_+(b)} \cong \widetilde{N_{(b)}}$$

   $$\left(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}\right)\Big|_{D_+(b)} \cong \widetilde{M}|_{D_+(b)} \otimes_{\mathcal{O}_X|_{D_+(b)}} \widetilde{N}|_{D_+(b)} \cong \widetilde{M_{(b)} \otimes_{S_{(b)}} N_{(b)}}$$

   and

   $$\widetilde{M \otimes_S N}|_{D_+(b)} \cong \widetilde{(M \otimes_S N)_{(b)}}.$$

   Therefore we have to define maps $\vartheta_b\colon M_{(b)} \otimes_{S_{(b)}} N_{(b)} \to (M \otimes_S N)_{(b)}$. For this we use the natural map given by

   $$\frac{m}{b^r} \otimes \frac{n}{b^{r'}} \mapsto \frac{m \otimes n}{b^{r+r'}}.$$

   The maps $\vartheta_b$ are compatible and thus glue to define $\phi$.

   If $b \in S_1$, then $\vartheta_b$ is an isomorphism. Hence $\phi|_{D_+(b)}$ is an isomorphism for every $b \in S_1$ by remark 2.71. But since $S_1$ generates $S$ as an $S_0$-algebra, the $D_+(b)$ with $b \in S_1$ cover $X$. Therefore $\phi$ is an isomorphism.

2. Using the identities $M(n) \cong M \otimes_S S(n)$ and $S(n) \otimes_S S(m) \cong S(m+n)$ this is a direct consequence of part 1.

3. Pick $c \in T$ homogeneous and of positive degree. Pick $b \in S$ homogeneous of the same degree with $\alpha(b) = c$. One easily sees that $f^{-1}D_+(b) = D_+(c)$.

   We will define a morphism $\psi\colon f^*\widetilde{M} \to \widetilde{M \otimes_S T}$. Similar to the first part, we define it on the open sets $D_+(b)$. We have (using theorem 2.70 for the last step)

   $$\left(f^*\widetilde{M}\right)\Big|_{D_+(c)} = f^*\left(\widetilde{M}|_{D_+(b)}\right) = f^*\left(\widetilde{M_{(b)}}\right) \cong \widetilde{M_{(b)} \otimes_{S_{(b)}} T_{(c)}}.$$

   Also $\widetilde{M \otimes_S T}|_{D_+(c)} \cong \widetilde{(M \otimes_S T)_{(c)}}$. Together with the homomorphism

   $$\mu\colon M_{(b)} \otimes_{S_{(b)}} T_{(c)} \to (M \otimes_S T)_{(c)}, \qquad \frac{m}{b^r} \otimes \frac{t}{c^{r'}} \mapsto \frac{m \otimes t}{c^{r+r'}}$$

   this defines $\psi$. If $c \in T_1$, then $\mu$ is an ismorphism. Hence $\psi$ is an isomorphism over $D_+(c)$ and so, like above, it is a global isomorphism.

   [The second statement was not proved in the lecture.] □

*Remark* 2.93. Under some mild conditions, if $\mathcal{F}$ is a quasi-coherent sheaf on $X = \operatorname{Proj} S$, then $\mathcal{F} = \widetilde{\Gamma_*(\mathcal{F})}$, where $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}(n)(X)$.

## 2.6. Cartier Divisors[1]

**Motivation** Let $X$ be a smooth variety over an algebraically closed field $k$. A subvariety $D \subseteq X$ is called a *prime divisor* if $\dim D = \dim X - 1$. A *divisor* is a linear combination $D = \sum n_i D_i$ with $n_i \in \mathbb{Z}$ and the $D_i$ prime divisors.

For every $x \in X$ there exists a rational function $f$ on $X$ such that $D|_U = (f)|_U$ for some open neighborhood $U$ of $x$. So a divisor is given by a collection $(U_i, f_i)$ (where the $f_i$ have to satisfy some compatibility conditions). To extend the notion of divisor to arbitrary schemes, we use this last property for the definition of a divisor. In order to avoid some technicalities we will assume the our schemes are integral. For a more general definition see [Har77, Section II.6] or [Liu02, Section 7.1].

Note that if $X$ is an integral scheme with function field $K(X)$, then for every open subset $U \subseteq X$ there is an injection $\mathcal{O}_X(U) \hookrightarrow K(X)$ [Liu02, Proposition 2.4.18].

**Definition 2.94.** Let $X$ be an integral scheme. A *Cartier divisor* $D$ on $X$ is an equivalence class of systems $(U_i, f_i)$, where the open sets $U_i$ cover $X$ and $f_i \in K(X)$ such that for all $i, j$

$$\frac{f_i}{f_j} \in \mathcal{O}_X^*(U_i \cap U_j) = \{\text{invertible elements of } \mathcal{O}_X(U_i \cap U_j)\} \subseteq K(X)$$

where two systems $(U_i, f_i)$ and $(V_\alpha, g_\alpha)$ are equivalent if for all $i, \alpha$

$$\frac{f_i}{g_\alpha} \in \mathcal{O}_X^*(U_i \cap V_\alpha).$$

**Definition 2.95.** Let $D, D'$ be two Cartier divisors on an integral scheme $X$, given by $(U_i, f_i)$ and $(U_j', f_j')$. Their *sum* $D + D'$ is defined to be the system $(U_i \cap U_j', f_i f_j')$ (as $X$ is integral the intersection is always nonempty). The *inverse* $-D$ of $D$ is given by $(U_i, f_i^{-1})$. Divisors of the form $(X, f)$ for some $f \in K(X)$ are called *principal*. The *group of Cartier divisors* is

$$\mathrm{Div}(X) = \frac{(\text{group of all Cartier divisor}, +)}{\text{principal divisors}}.$$

*Remark** 2.96. Two divisor $D$, $D'$ are called *linearly equivalent*, denoted $D \sim D'$, if their difference is principal. So $\mathrm{Div}(X)$ is the group of all Cartier divisors modulo linear equivalence. Also note that many authors (e.g. [Har77, Liu02]) denote the full group of divisors by $\mathrm{Div}(X)$ and the group of divisors modulo linear equivalence by $\mathrm{CaCl}(X)$ and call the latter the (Cartier) divisor class group.

**Definition 2.97.** Let $D$ be a Cartier divisor on an integral scheme $X$ given by $(U_i, f_i)$. To $D$ we associate the sheaf $\mathcal{O}_X(D)$ given by (for each $U \subseteq X$ open)

$$\mathcal{O}_X(D)(U) = \{h \in K(X) : hf_i \in \mathcal{O}_X(U_i \cap U) \subseteq K(X) \text{ for all } i\}.$$

Note that this is well-defined: Suppose that $D$ is also given by the system $(V_\alpha, g_\alpha)$. For $h \in \mathcal{O}_X(D)(U)$ we have $hf_i \in \mathcal{O}_X(U_i \cap U)$, so

$$hg_\alpha|_{U \cap U_i \cap V_\alpha} = \underbrace{hf_i}_{\in \mathcal{O}_X(U \cap U_i \cap V_\alpha)} \cdot \underbrace{\frac{g_\alpha}{f_i}}_{\in \mathcal{O}_X(U \cap U_i \cap V_\alpha)} \in \mathcal{O}_X(U \cap U_i \cap V_\alpha).$$

Since the sets $U \cap U_i \cap V_\alpha$ cover $U \cap V_\alpha$, the sheaf condition implies $hg_\alpha \in \mathcal{O}_X(U \cap V_\alpha)$. Therefore if $h \in \mathcal{O}_X(D)(U)$ as defined by $(U_i, f_i)$, then $h$ is also in $\mathcal{O}_X(D)(U)$ defined by $(V_\alpha, g_\alpha)$. By a symmetric argument the converse inclusion holds too.

[1]The proofs in this section are non-examinable.

**Theorem 2.98.** *Let $X$ be an integral scheme and let $D, D'$ be Cartier divisors on $X$.*

1. *$\mathcal{O}_X(D)$ is an invertible sheaf.*
2. *$\mathcal{O}_X(D+D') \cong \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D')$.*
3. *$\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^{-1}$.*
4. *$D - D'$ is principal if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$.*

*Therefore the map $D \mapsto \mathcal{O}_X(D)$ induces an injective homomorphism* $\mathrm{Div}(X) \to \mathrm{Pic}(X)$.

*Proof.*

1. Let $D$ be given by $(U_i, f_i)$. Then $\mathcal{O}_X(D)|_{U_i} = \mathcal{O}_{U_i} \cdot \frac{1}{f_i}$ as modules over $\mathcal{O}_{U_i}$. So $\mathcal{O}_X(D)|_{U_i} \cong \mathcal{O}|_{U_i}$ as $\mathcal{O}_{U_i}$-modules. Thus $\mathcal{O}_X(D)$ is invertible.
2. Define a morphism
$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \xrightarrow{\psi} \mathcal{O}_X(D+D')$$
in the following way: For any open set $V \subseteq X$ define
$$\mathcal{O}_X(D)(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(D')(V) \to \mathcal{O}_X(D+D')(V), \quad h \otimes h' \mapsto hh'.$$
We can assume that $D$ and $D'$ are given by $(U_i,\, f_i)$ and $(U_i,\, f_i')$ respectively. So $D + D'$ is given by $(U_i,\, f_i f_i')$. Hence if $hf_i$ and $h'f_i'$ are in $\mathcal{O}_X(V \cap U_i)$ for all $i$, the $hh'f_if_i' \in \mathcal{O}_X(V \cap U_i)$, so $hh' \in \mathcal{O}_X(D+D')(V)$, i.e. $\psi$ is well-defined.

   Now check that it is an isomorphism.
3. "exercise or ignore" □

## 2.7. Differential Forms[2]

**Definition 2.99.** Let $A$ be a ring, $B$ an $A$-algebra and $M$ a $B$-module. An *$A$-derivation from $B$ into $M$* is a map $\mathrm{d}\colon B \to M$ with

1. $\mathrm{d}(b + b') = \mathrm{d}b + \mathrm{d}b'$ for all $b, b' \in B$;
2. $\mathrm{d}(bb') = b\,\mathrm{d}b' + b'\,\mathrm{d}b$ for all $b, b' \in B$;
3. $\mathrm{d}a = 0$ for all $a \in A$.

The *module of relative differential forms of $B$ over $A$* is a $B$-module $\Omega_{B/A}$ together with an $A$-derivation $\mathrm{d}\colon B \to \Omega_{B/A}$ such that for any $B$-module $M$ and $A$-derivation $\mathrm{d}'\colon B \to M$ there exists a unique $B$-module homomorphism $f\colon \Omega_{B/A} \to M$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[rr, "\mathrm{d}"] \arrow[dr, "\mathrm{d}'"'] & & \Omega_{B/A} \arrow[dl, dashed, "\exists! f"] \\ & M & \end{tikzcd}$$

**Proposition 2.100.** *The pair $(\Omega_{B/A}, \mathrm{d})$ exists and is unique up to unique isomorphism. It can be constructed in the following way: $\Omega_{B/A}$ is the free $B$-module generated by symbols of the form $\mathrm{d}f$, $f \in B$, modulo the relations*

1. *$\mathrm{d}(f+g) = \mathrm{d}f + \mathrm{d}g$ for all $f, g \in B$;*
2. *$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$ for all $f, g \in B$;*

[2]The proofs in this section are non-examinable.

*3.* $\mathrm{d}a = 0$ *for all* $a \in A$.

*Example* 2.101. For $B = A[t_1, \dots, t_n]$, we get $\Omega_{B/A} \cong B^{\oplus n}$: every element of $\Omega_{B/A}$ can be written as a sum $\sum_{i=1}^n f_i \,\mathrm{d}t_i$ with $f_i \in B$ in a unique way (this needs some work to prove).

**Definition 2.102.** Let $f\colon X \to Y$ be a morphism of affine schemes, say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. The *sheaf of relative differential forms* on $X$ over $Y$ is $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$.

More generally, if $f\colon X \to Y$ is any morphism of schemes, the *sheaf of relative differential forms* on $X$ over $Y$, again denoted $\Omega_{X/Y}$ is defined as follows: Cover $Y$ with open affine schemes $U_i$ and cover each $f^{-1}U_i$ with open affine schemes $V_{i,\alpha}$. Then for each pair $i, \alpha$ the restriction of $f$ gives the sheaf $\Omega_{V_{i,\alpha}/U_i}$. These sheaves can be glued together to form the sheaf $\Omega_{X/Y}$ on $X$. [Liu02, Proposition 6.1.17]

*Example* 2.103. If $X = \mathbb{A}^n_A$, $Y = \operatorname{Spec} A$ and $B = A[t_1, \dots, t_n]$, then $\Omega_{X/Y} = \widetilde{\Omega_{B/A}} = \widetilde{B}^{\oplus n}$.

**Theorem 2.104.** *Let* $f\colon X \to Y$ *and* $g\colon S \to Y$ *be morphisms of schemes and let* $p\colon X \times_Y S \to X$ *be the projection. Then*

$$p^*\Omega_{X/Y} = \Omega_{X\times_Y S/S}.$$

**Theorem 2.105.** *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are morphisms of schemes, then the sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

*is exact.*

**Theorem 2.106.** *Suppose that* $A$ *is a ring,* $X = \mathbb{P}^n_A$ *and* $Y = \operatorname{Spec} A$*. Then there is an exact sequence*

$$0 \to \Omega_{X/Y} \to \bigoplus_{i=0}^n \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0.$$

*Proof.* Let $S = A[t_0, \dots, t_n]$ and let $T$ be the $S$-module $S(-1)^{\oplus(n+1)}$ with free basis $e_0, \dots, e_n$. Further let $\alpha\colon T \to S$ be the (graded) $S$-module homomorphism given by $e_i \mapsto t_i$ and let $M$ be its kernel. This gives an exact sequence of $S$-module

$$0 \to M \to T \to S.$$

Passing to sheaves, we get an exact sequence

$$0 \to \widetilde{M} \to \widetilde{T} \to \widetilde{S}.$$

By definition, $\widetilde{T} = \mathcal{O}_X(-1)^{\oplus(n+1)}$ and $\widetilde{S} = \mathcal{O}_X$. The map $\alpha\colon T \to S$ is not surjective, but one can show that the corresponding map of sheaves is surjective. So the above sequence is really the exact sequence

$$0 \to \widetilde{M} \to \mathcal{O}_X(-1)^{\oplus(n+1)} \to \mathcal{O}_X \to 0.$$

We will now show that $\widetilde{M} \cong \Omega_{X/Y}$. In order to do this, we cover $X$ by the open affines $U_i = D_+(t_i)$ and define $\phi_i\colon \Omega_{X/Y}|_{U_i} \to \widetilde{M}|_{U_i}$ as follows: Let $B_i = S_{(t_i)} = A[\frac{t_0}{t_i}, \dots, \frac{t_n}{t_i}]$. Then $\Omega_{X/Y}|_{U_i} = \Omega_{U_i/Y} = \widetilde{\Omega_{B_i/A}}$ and $\Omega_{B_i/A}$ is generated (as a $B_i$-module) by $\mathrm{d}(\frac{t_k}{t_i})$ $(k = 0, \dots, n)$. We define a homomorphism $h_i\colon \Omega_{B_i/A} \to M_{(t_i)}$ by $\mathrm{d}(\frac{t_k}{t_i}) \mapsto \frac{1}{t_i^2}(t_i e_k - t_k e_i)$. ($t_i e_k - t_k e_i \in \ker\alpha$ and the $\frac{1}{t_i^2}$ is needed to obtain an element of degree 0.) It turns out that $h_i$ is an isomorphism. We define $\phi_i$ to be the morphism of sheaves induced by $h_i$. So $\phi_i$ is also an isomorphism.

To obtain an isomorphism $\Omega_{X/Y} \to \widetilde{M}$ we will glue the $\phi_i$: Because of $\frac{t_k}{t_i} = \frac{t_k}{t_j}\frac{t_j}{t_i}$ on $U_i \cap U_j$ we have

$$\mathrm{d}(\frac{t_k}{t_i}) = \frac{t_k}{t_j}\,\mathrm{d}(\frac{t_j}{t_i}) + \frac{t_j}{t_i}\,\mathrm{d}(\frac{t_k}{t_j}),$$

which one can use to show that the $\phi_i$ are compatible. □

*Remark* 2.107. Let $X$ is a variety over an algebraically closed field $k$ (i.e. an integral and separated scheme of finite type over $\operatorname{Spec} k$). Then $X$ is smooth (i.e. all local rings $\mathcal{O}_{X,x}$ are regular) if and only if $\Omega_{X/\operatorname{Spec} k}$ is locally free (i.e. locally it is isomorphic to a direct sum of copies of $\mathcal{O}_X$).

If $X$ is a smooth variety over $k$, the *tangent sheaf* to $X$ is defined as $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$. Further the sheaf $\omega_X = \Lambda^{\dim X}\Omega_{X/\operatorname{Spec} k}$ is called the *canonical sheaf*. It is invertible and the associated divisor is called the *canonical divisor*. It plays an important role in the Riemann-Roch theorem (see 3.40).

# 3. Cohomology

## 3.1. Cohomology of Sheaves

**Theorem 3.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\mathfrak{M}(X)$ of $\mathcal{O}_X$-modules on $X$ has enough injectives.*

*Proof.* Let $\mathcal{F}$ be any $\mathcal{O}_X$-module. We need to find an injective $\mathcal{O}_X$-module $I$ and an injective morphism $\mathcal{F} \to I$.

For each $x \in X$, the stalk $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module. By commutative algebra [Eis95, Corollary A3.9], there exists a monomorphism of $\mathcal{O}_{X,x}$-modules $\mathcal{F}_x \to I_x$ such that $I_x$ is injective. We can consider $I_x$ as a sheaf on the one-point space $\{x\}$. Let $f_x \colon \{x\} \to X$ be the inclusion map. Then $f_{x*}I_x$ is an $\mathcal{O}_X$-module. We set

$$I = \prod_{x \in X} f_{x*}I_x.$$

We will first show that there is an injective morphism $\mathcal{F} \to I$. Consider any $\mathcal{O}_X$-module $\mathcal{G}$. By the universal property of direct products,

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, I) \cong \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_{x*}I_x).$$

We will show that for each $x \in X$, we have $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_{x*}I_x) \cong \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$: If $\phi \colon \mathcal{G} \to f_{x*}I_x$ is a morphism of $\mathcal{O}_X$-modules, then $\phi_x \colon \mathcal{G}_x \to I_x$ is in $\operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$. Conversely, if we are given an $\mathcal{O}_{X,x}$-module homomorphism $\mathcal{G}_x \to I_x$, then then for any open $U \subseteq X$ with $x \in U$ we have the composition

$$\mathcal{G}(U) \to \mathcal{G}_x \to I_x = (f_{x*}I_x)(U)$$

(and for $x \notin U$, $(f_{x*}I_x)(U) = 0$, so that in this case there is a unique map $\mathcal{G}(U) \to (f_{x*}I_x)(U)$) and these give a morphism $\mathcal{G} \to (f_{x*}I_x)(U)$.

In particular, by construction of $I$, for each point $x \in X$ we have an injection $\mathcal{F}_x \to I_x$ and these gives a morphism $\mathcal{F} \to I$, which has to be injective as it is injective on stalks.

Finally we will show that $I$ is injective: Suppose we are given an injective morphism $\mathcal{G} \to \mathcal{H}$ of $\mathcal{O}_X$-modules and a morphism $\mathcal{G} \to I$. By the above discussion this situation corresponds to commutative diagrams (with the first row exact)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}_x \arrow[r] \arrow[d] & \mathcal{H}_x \arrow[dl, dashed, "\exists"] \\ & I_x & \end{tikzcd}$$

for each $x \in X$. Since $I_x$ is injective, there exists a homomorphism $\mathcal{H}_x \to I_x$ fitting into the above diagram. Again by the above discussion, these homomorphism give a morphism $\mathcal{H} \to I$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[dl] \\ & I & \end{tikzcd}$$

commutes. Now we can inductively construct an injective resolution for $\mathcal{F}$ as follows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r, hook] & I \arrow[rr, dashed, bend left] \arrow[dr] & & I^1 \arrow[rr, dashed, bend left] \arrow[dr] & & I^2 & \cdots \\ & & & I^0/\mathcal{F} \arrow[ur, hook] \arrow[dr] & & I^1/(I^0/\mathcal{F}) \arrow[ur, hook] \arrow[dr] & & \\ & & & & 0 & & 0 & \end{tikzcd}$$

□

**Corollary 3.2.** *Let $X$ be a topological space. Then the category $\mathfrak{Sh}(X)$ of sheaves of Abelian groups on $X$ has enough injectives.*

*Proof.* We can consider $X$ as a ringed space by setting $\mathcal{O}_X$ to be the constant sheaf defined by $\mathbb{Z}$. Then $\mathfrak{Sh}(X) = \mathfrak{M}(X)$ and we can apply the preceding theorem. □

**Definition 3.3.** Let $X$ be a topological space and let $F\colon \mathfrak{Sh}(X) \to \mathfrak{Ab}$ be the global sections functor $\mathcal{F} \mapsto \mathcal{F}(X)$. Then the *$i$-th cohomology group of* $\mathcal{F}$ is $H^i(X, \mathcal{F}) = R^i F(\mathcal{F})$.

**Definition 3.4.** Let $X$ be a topological space. A sheaf $\mathcal{F}$ on $X$ is called *flasque* if for every open subset $U \subseteq X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is a surjection.

*Remark** 3.5. If $\mathcal{F}$ is flasque, then for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

**Theorem 3.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then every injective $I \in \mathfrak{M}(X)$ is flasque.*

*Proof.* Pick any open subset $U$ of $X$. We have to show that the restriction $I(X) \to I(U)$ is surjective. Define a presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on $X$ by

$$\mathcal{F}(W) = \begin{cases} 0, & \text{if } W \not\subseteq U \\ \mathcal{O}_X(W), & \text{if } W \subseteq U \end{cases}$$

for every open $W \subseteq X$. Pick any element $t \in I(U)$. We have to find $s \in I(X)$ with $s|_U = t$. In order to do this define a morphism $\phi^+\colon \mathcal{F}^+ \to I$ by

$$\begin{aligned} \phi_W\colon \mathcal{F}(W) &\to I(W) \\ 1 &\mapsto \begin{cases} 0, & \text{if } W \not\subseteq U \\ t|_W, & \text{if } W \subseteq U \end{cases}. \end{aligned}$$

Since $I$ is injective, we can find $\psi\colon \mathcal{O}_X \to I$ fitting into the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}^+ \arrow[r, hook] \arrow[d, dashed, "\phi^+"'] & \mathcal{O}_X \arrow[dl, dashed, "\psi"] \\ & I & \end{tikzcd}$$

Put $s = \psi_X(1) \in I(X)$. Then

$$s|_U = \psi_X(1)|_U = \psi_U(1|_U) = \psi_U(1) = \phi_U^+(1) = t. \qquad \square$$

**Lemma 3.7.** *Let $X$ be a topological space and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

*a short exact sequence of sheaves with $\mathcal{F}$ flasque. Then the map $\mathcal{G}(X) \to \mathcal{H}(X)$ on global sections is surjective.*

*Proof.* Denote the map $\mathcal{G} \to \mathcal{H}$ by $\alpha$. Note that locally $\alpha$ is surjective and we have to prove that it is surjective on global sections. Pick any $t \in \mathcal{H}(X)$. We have to find a section $s \in \mathcal{G}(X)$ that maps to $t$. Let $\mathfrak{S}$ be the set of all pairs $(U, s)$ such that $s \in \mathcal{G}(U)$ maps to $t|_U$. Define a partial order on $\mathfrak{S}$ by setting $(U_1, s_1) \leq (U_2, s_2)$ if $U_1 \subseteq U_2$ and $s_2|_{U_1} = s_1$. By the sheaf condition every chain in $\mathfrak{S}$ is bounded. Thus Zorn's lemma provides a maximal element $(U', s')$ of $\mathfrak{S}$.

Suppose $U' \neq X$. Let $x \in X \setminus U'$. Then, by local surjectivity, there exists an open neighborhood $V$ of $x$ and a section $\widetilde{s} \in \mathcal{G}(V)$ such that $(V, \widetilde{s}) \in \mathfrak{S}$. Then

$$s'|_{U' \cap V} - \widetilde{s}|_{U' \cap V} \in \ker\left(\mathcal{G}(U' \cap V) \to \mathcal{H}(U' \cap V)\right) = \mathcal{F}(U' \cap V).$$

Since $\mathcal{F}$ is flasque there exists a global section $r \in \mathcal{F}(X) \subseteq I(X)$ such that $r|_{U' \cap V} = s'|_{U' \cap V} - \widetilde{s}|_{U' \cap V}$. Then $(r|_V + \widetilde{s})$ and $s'$ restrict to the same section in $\mathcal{G}(U' \cap V)$. Hence they glue to give an element $(U' \cup V, s)$ of $\mathfrak{S}$. This is a contradiction to the maximality of $(U', s')$. Therefore $U' = X$ and $s'$ maps to $t$. So $\alpha_X$ is surjective. $\square$

**Lemma 3.8.** *Let $X$ be a topological space and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

*a short exact sequence of sheaves with $\mathcal{F}$ and $\mathcal{G}$ flasque. Then $\mathcal{H}$ is also flasque.*

*Proof.* Applying Lemma 3.7 to the sheaves restricted to any open $U \subseteq X$ we see that all sequences

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

are exact. For any open $U \subseteq X$ consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{G}(X) & \longrightarrow & \mathcal{H}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) & \longrightarrow & \mathcal{H}(U) & \longrightarrow & 0
\end{array}$$

where the rows are exact and the vertical arrows are the restriction maps. Applying the four lemma we see that the restriction map $\mathcal{H}(X) \to \mathcal{H}(U)$ is also surjective, so that $\mathcal{H}$ is flasque. $\square$

**Theorem 3.9.** *Let $X$ be a topological space and $\mathcal{F}$ a flasque sheaf on $X$. Then the cohomology groups $H^i(X, \mathcal{F})$ are zero for all $i \geq 1$. In other words, every flasque sheaf is acyclic with respect to the global sections functor on $\mathfrak{Sh}(X)$.*

*Proof.* Let $I$ be the first step in an injective resolution if $\mathcal{F}$ and set $\mathcal{G} = I/\mathcal{F}$ so that we have the short exact sequence

$$0 \to \mathcal{F} \to I \to \mathcal{G} \to 0. \tag{3.1}$$

Note that $\mathcal{F}$ is flasque by assumption and $I$ is flasque by theorem 3.6 hence $\mathcal{G}$ is flasque by the lemma.

We apply the long exact sequence of (3.1):

$$
\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{F}) \arrow[r] & H^0(X, I) \arrow[r] & H^0(X, \mathcal{G}) \arrow[dll, out=-10, in=170] \\
& H^1(X, \mathcal{F}) \arrow[r] & H^1(X, I) \arrow[r] & H^1(X, \mathcal{G}) \arrow[dll, out=-10, in=170] \\
& H^2(X, \mathcal{F}) \arrow[r, dashed] & {}
\end{tikzcd}
$$

Since $I$ is injective, $H^i(X, I) = 0$ for $i \geq 1$. Also the sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, I) \to H^0(X, \mathcal{G}) \to 0$$

is exact by Lemma 3.7 and hence $H^1(X, \mathcal{F}) = 0$. The same argument applied to the flasque sheaf $\mathcal{G}$ shows that $H^1(X, \mathcal{G}) = 0$. Now an easy induction shows that $H^i(X, \mathcal{F})$ (and hence also $H^i(X, \mathcal{G})$) is zero for $i \geq 0$. $\square$

**Corollary 3.10.** *Let $X$ be a topological space and $\mathcal{F}$ a sheaf on $X$. Then cohomology can be computed using "flasque resolutions": For any exact sequence*

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots,$$

*with all $\mathcal{I}^i$ flasque, the cohomology groups of*

$$0 \to \mathcal{I}^0(X) \to \mathcal{I}^1(X) \to \mathcal{I}^2(X) \to \cdots$$

*are isomorphic to $H^i(X, \mathcal{F})$.*

*Proof.* This immediately follows from the last theorem and the general fact that derived functors can be computed using an "acyclic resolution" (theorem 1.43). $\square$

**Corollary 3.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of the global sections functor $\mathfrak{M}(X) \to \mathfrak{Ab}$, $\mathcal{F} \to \mathcal{F}(X)$ coincide with the cohomology functors $H^i(X, -)$ (computed with respect to $\mathfrak{Sh}(X)$).*

*Proof.* To compute the derived functors of the global sections functor of an $\mathcal{O}_X$-module $\mathcal{F}$, we have to chose an injective resolution of $\mathcal{F}$ in $\mathfrak{M}(X)$. By theorem 3.6, injective $\mathcal{O}_X$-modules are flasque and by the corollary 3.10 a flasque resolution gives the cohomology groups $H^i(X, \mathcal{F})$. $\square$

**Definition 3.12.** Let $X$ be a topological space. The *dimension* of $X$ is the supremum of the all integers $n$ such that there is a chain $\emptyset \neq X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$ of irreducible closed subsets of $X$.

[Har77, Theorem III.2.7]

**Theorem 3.13.** *Suppose $X$ is a Noetherian topological space of dimension $d$. Then for every sheaf $\mathcal{F}$ on $X$ the cohomology groups $H^i(X, \mathcal{F})$ vanish for all $i > d$.*

*Proof.* Omitted. $\square$

[Har77, Proposition III.3.4]

**Lemma 3.14.** *Let $A$ be a Noetherian ring and $I$ and injective $A$-module. Then $\widetilde{I}$ is a flasque sheaf on* Spec $A$.

*Proof.* Omitted. $\square$

**Theorem 3.15.** *Let $X$ be a Noetherian scheme. Then the following are equivalent:*

1. *$X$ is affine;*

2. $H^i(X, \mathcal{F}) = 0$ *for all quasi-coherent sheaves* $\mathcal{F}$ *on* $X$ *and all* $i \geq 1$*;*

3. $H^1(X, \mathcal{F}) = 0$ *for all coherent ideal sheaves* $\mathcal{F}$ *on* $X$*.*

*Proof.*

**(1) implies (2)** Since $X$ is affine, we can assume $X = \operatorname{Spec} A$ with $A$ Noetherian. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. By theorem 2.78, $\mathcal{F} = \widetilde{M}$ for some $A$-module $M$. Any injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

of $M$ gives an exact sequence (by remark 2.72)

$$0 \to \mathcal{F} = \widetilde{M} \to \widetilde{I^0} \to \widetilde{I^1} \to \widetilde{I^2} \to \cdots .$$

By the preceding lemma this sequence is a flasque resolution of $X$; so we can use it to calculate the cohomology groups. But applying the global sections functor to this sequence recovers the original sequence of modules which is exact. So the cohomology groups $H^i(X, \mathcal{F})$ vanish for $i \geq 1$.

**(2) implies (3)** trivial

**(3) implies (1)** This is the hardest part. We are going to use the criterion of exercise A.27. In order to do so we need to construct elements $b \in \mathcal{O}_X(X)$ such that $D(b) = \{x \in X : b \notin \mathfrak{m}_x\}$ is an open affine subscheme.

Pick any closed point $x \in X$ and any open affine neighborhood $U$ of $x$. Set $Y = X \setminus U$. Let $I_Y$ be a quasi-coherent ideal sheaf corresponding to some subscheme structure on $Y$. We can also consider $\{x\}$ as the closed subscheme of $X$ given by $\operatorname{Spec} k(x)$ (cf. exercise A.9). Further, let $I_{Y \cup \{x\}}$ be the ideal sheaf corresponding to the subscheme structure on the closed subset $Y \cup \{x\}$. Since $X$ is Noetherian both $I_Y$ and $I_{Y \cup \{x\}}$ are coherent. This gives a short exact sequence

$$0 \to I_{Y \cup \{x\}} \to I_Y \to k(x) \to 0$$

where $k(x)$ is identified with the skyscraper sheaf at $x$ given by $k(x)$. The long exact sequence of this sequence is

$$0 \to I_{Y \cup \{x\}}(X) \to I_Y(X) \xrightarrow{\alpha} k(x) \to H^1(X, I_{Y \cup \{x\}}) \to \cdots .$$

By assumption $H^1(X, I_{Y \cup \{x\}}) = 0$, so $\alpha$ is surjective. Hence there exists $b \in I_Y(X) \subseteq \mathcal{O}_X(X)$ with $\alpha(b) = 1$. Thus $x \in D(b)$. Also, since $b \in I_Y(X)$, $D(b) \cap Y = \emptyset$, i.e. $D(b) \subseteq U$. But $U$ is affine, so $D(b) = D(b|_U) \subseteq U$ is an open affine subset.

In this way, we can cover $X$ with open affine sets of the form $D(b)$. Since $X$ is Noetherian, it can be covered by only finitely many, say $D(b_1), \ldots, D(b_n)$. We have to show that $b_1, \ldots, b_n$ generate $\mathcal{O}_X(X)$.

Define a morphism $\phi\colon \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X$ by $\phi_W\left((\delta_{ij})_{j=1}^n\right) = b_i|_W$ for all open subsets $W \subseteq X$. For affine subsets $U$ of $X$, $b_1|_U, \ldots, b_n|_U$ generate $\mathcal{O}_X(U)$, so $\phi$ is surjective. We have to show that $\phi_X$ is surjective.

Set $\mathcal{F} = \ker \phi$, so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X \to 0. \tag{3.2}$$

Consider the filtrations

$$\mathcal{G}_0 = 0 \subseteq \mathcal{G}_1 = \mathcal{O}_X \oplus 0 \oplus \cdots \oplus 0 \subseteq \mathcal{G}_2 = \mathcal{O}_X \oplus \mathcal{O}_X \oplus 0 \oplus \cdots \oplus 0 \subseteq \cdots \subseteq \mathcal{G}_n = \mathcal{O}_X^{\oplus n}$$

and

$$\mathcal{F}_0 = 0 \subseteq \mathcal{F}_1 = \mathcal{G}_1 \cap \mathcal{F} \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{G}_n \cap \mathcal{F}.$$

This gives injections

$$\frac{\mathcal{F}_{i+1}}{\mathcal{F}_i} = \frac{\mathcal{F} \cap G_{i+1}}{\mathcal{F} \cap \mathcal{G}_i} \hookrightarrow \frac{\mathcal{G}_{i+1}}{\mathcal{G}_i} \cong \mathcal{O}_X,$$

so that $\mathcal{F}_{i+1}/\mathcal{F}_i$ is an ideal sheaf of $X$. As $X$ is Noetherian, $\mathcal{F}_{i+1}/\mathcal{F}_i$ is coherent. So by assumption, $H^1(X, \mathcal{F}_{i+1}/\mathcal{F}_i) = 0$. From the long exact sequence of

$$0 \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \mathcal{F}_{i+1}/\mathcal{F}_i \to 0$$

and $\mathcal{F}_0 = 0$ we can inductively show that $H^1(X, \mathcal{F}_i) = 0$. In particular, $H^1(X, \mathcal{F}) = 0$. Now the long exact sequence of (3.2) is

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{O}_X^{\oplus n}) \xrightarrow{\phi_X} H^0(X, \mathcal{O}_X) \to H^1(X, \mathcal{F}) = 0.$$

Therefore $\phi_X$ is surjective as required. □

## 3.2. Čech Cohomology

**Definition 3.16.** Let $X$ be a topological space. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of $X$ indexed by a well-ordered set $I$. Let $\mathcal{F}$ be a sheaf on $X$. We will write $U_{i_0,\ldots,i_p}$ for $U_{i_0} \cap \cdots \cap U_{i_p}$.

- For $p \geq 0$ define
$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p}).$$
By convention, the empty product is 0, i.e. if $|I| < p + 1$, then $C^p(\mathfrak{U}, \mathcal{F}) = 0$.

- For $p \geq 0$ define a homomorphism
$$d^p : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$$
by sending $(s_{i_0,\ldots,i_p})_{i_0 < \cdots < i_p}$ to $(t_{j_0,\ldots,j_{p+1}})_{j_0 < \cdots < i_{p+1}}$ with
$$t_{j_0,\ldots,j_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \, s_{j_0,\ldots,\widehat{j_k},\ldots,j_{p+1}}\Big|_{U_{j_0,\ldots,j_{p+1}}}.$$

- It is easy to check that $d^{p+1} \circ d^p = 0$. The complex
$$0 \xrightarrow{d^{-1}} C^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^1} C^2(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^2} \cdots$$
is called *Čech complex.*

- The $p$th *Čech cohomology group of* $\mathcal{F}$ *with respect to the covering* $\mathfrak{U}$ is
$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = \frac{\ker d^p}{\operatorname{im} d^{p-1}}.$$

**Theorem 3.17.** *Using the notation of the definition,*

$$\check{H}^0(\mathfrak{U}, \mathcal{F}) \cong H^0(X, \mathcal{F}) = \mathcal{F}(X).$$

*Proof.* By definition,

$$C^0(\mathfrak{U}, \mathcal{F}) = \prod_{i\in I} \mathcal{F}(U_i),$$

$$d^0(s_i)_{i\in I} = \left(s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j}\right)_{i<j}.$$

Therefore

$$\check{H}^0(\mathfrak{U}, \mathcal{F}) = \ker d^0 = \left\{(s_i) \in C^0(\mathfrak{U}, \mathcal{F}) : s_j|_{U_i\cap U_j} = s_i|_{U_i\cap U_j} \forall i,j\right\}.$$

By the sheaf condition such a family $(s_i)$ uniquely determines a section $s \in \mathcal{F}(X)$, so $\ker d^0 \cong \mathcal{F}(X)$. $\square$

*Example* 3.18. Let $X = \mathbb{P}^1_K = \operatorname{Proj} K[t_0, t_1]$, where $K$ is a field. Let $\mathfrak{U} = (U_0, U_1)$ with $U_i = D_+(t_i)$. We will calculate $\check{H}^\bullet(\mathfrak{U}, \mathcal{O}_{\mathbb{P}^1_K})$. The intersection $U_0 \cap U_1$ is equal to $D_+(t_0t_1)$, so by theorem 2.44 we have

$$\begin{aligned} C^0(\mathfrak{U}, \mathcal{O}_X) =& \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2) = K[t_0, t_1]_{(t_0)} \oplus K[t_0, t_1]_{(t_1)} \\ C^1(\mathfrak{U}, \mathcal{O}_X) =& \mathcal{O}_X(U_0 \cap U_1) = K[t_0, t_1]_{(t_0t_1)} \end{aligned}$$

Since $X$ is integral we can consider all sections as elements of the function field $K(X) = K(t_0, t_1)$. The Čech complex looks like

$$\begin{array}{c} 0 \xrightarrow{d^{-1}} K[t_0, t_1]_{(t_0)} \oplus K[t_0, t_1]_{(t_1)} \xrightarrow{d^0} K[t_0, t_1]_{(t_0t_1)} \xrightarrow{d^1} 0 \\ (p, q) \longmapsto q - p \end{array}$$

First we want to calculate $\check{H}^0(\mathfrak{U}, \mathcal{O}_X) = \ker d^0 = \{(p, q) : p = q\}$. We can write $p = \frac{F}{t_0^{d_0}}$, $q = \frac{G}{t_1^{d_1}}$ with $\deg F = d_0$, $\deg G = d_1$. So we have $\frac{F}{t_0^{d_0}} = \frac{G}{t_1^{d_1}}$ in $K(t_0, t_1)$, i.e. $Ft_1^{d_1} = Gt_0^{d_0}$ in $K[t_0, t_1]$. The last ring is a UFD, so $t_0^{d_0}|F$ and $t_1^{d_1}|G$. Hence there exists $a_0, a_1 \in K$ such that $F = a_0t_0^{d_0}$ and $G = a_1t_1^{d_1}$. This gives $a_0 = p = q = a_1$, so that

$$\mathcal{O}_X(X) = H^0(X, \mathcal{O}_X) = \check{H}^0(\mathfrak{U}, \mathcal{O}_X) = \ker d^0 = K.$$

Next we need to calculate $\operatorname{im} d^0$. Now, $\operatorname{im} d^0$ is the sub-$K$-vector space of $K[t_0, t_1]_{(t_0t_1)}$ generated by all elements of $K[t_0, t_1]_{(t_0)}$ and $K[t_0, t_1]_{(t_1)}$. We want to show that this all of $K[t_0, t_1]_{(t_0t_1)}$. Pick any $r \in K[t_0, t_1]_{(t_0t_1)}$ and write $r = \frac{E}{(t_0t_1)^d}$ for some $E \in K[t_0, t_1]$ with $\deg E = 2d$. By linearity, we may assume that $E$ is a monomial, say $E = t_0^m t_1^n$. Because $m + n = 2d$, either $m \geq d$ or $n \geq d$. If $m \geq d$, then $r = \frac{t_0^m t_1^n}{(t_0t_1)^d} = \frac{t_0^{m-d}t_1^n}{t_1^d} \in K[t_0, t_1]_{(t_1)}$. Similarly for $n \geq d$. Thus $\operatorname{im} d^0 = K[t_0, t_1]_{(t_0t_1)}$ and $\check{H}^1(\mathfrak{U}, \mathcal{O}_X) = 0$.

Obviously $\check{H}^p(\mathfrak{U}, X) = 0$ for $p \geq 2$. $\square$

*Example* 3.19. Like in the last example, let $X = \mathbb{P}^1_K = \operatorname{Proj} K[t_0, t_1]$, where $K$ is a field and cover $X$ by $\mathfrak{U} = (U_0, U_1)$ with $U_i = D_+(t_i)$. We will calculate $\check{H}^\bullet(\mathfrak{U}, \Omega_{\mathbb{P}^1_K/\operatorname{Spec} K})$.

We have $U_0 \cong \operatorname{Spec} K[t_0, t_1]_{(t_0)} = \operatorname{Spec} K[\frac{t_1}{t_0}]$. Therefore

$$\Omega_{X/\operatorname{Spec} K}(U_0) = \Omega_{\mathcal{O}_X(U_0)/K} = K[t_0, t_1]_{(t_0)} \,\mathrm{d}\left(\frac{t_1}{t_0}\right).$$

Similarly,

$$\Omega_{X/\operatorname{Spec} K}(U_1) = \Omega_{\mathcal{O}_X(U_1)/K} = K[t_0, t_1]_{(t_1)} \,\mathrm{d}\left(\frac{t_0}{t_1}\right).$$

Further, from $U_1 \cap U_2 = D(\frac{t_0}{t_1}) \subseteq U_1 \cong A^1_K$,

$$\Omega_{X/\operatorname{Spec} K}(U_0 \cap U_1) = \left(\Omega_{X/\operatorname{Spec} K}(U_1)\right)_{\left(\frac{t_0}{t_1}\right)} = K\left[\frac{t_0}{t_1}, \frac{t_1}{t_0}\right] \mathrm{d}\left(\frac{t_0}{t_1}\right).$$

So the Čech complex is given by

$$0 \to K[t_0, t_1]_{(t_0)} \,\mathrm{d}\left(\frac{t_1}{t_0}\right) \oplus K[t_0, t_1]_{(t_1)} \,\mathrm{d}\left(\frac{t_0}{t_1}\right) \xrightarrow{d^0} K\left[\frac{t_0}{t_1}, \frac{t_1}{t_0}\right] \mathrm{d}\left(\frac{t_0}{t_1}\right). \to 0$$

$$d^0\left(p\,\mathrm{d}\left(\frac{t_1}{t_0}\right), q\,\mathrm{d}\left(\frac{t_0}{t_1}\right)\right) = q\,\mathrm{d}\left(\frac{t_0}{t_1}\right) - p\,\mathrm{d}\left(\frac{t_1}{t_0}\right).$$

On $U_1 \cap U_2$ we have $\frac{t_0}{t_1} \cdot \frac{t_1}{t_0} = 1$, so that $\frac{t_0}{t_1}\,\mathrm{d}\left(\frac{t_1}{t_0}\right) + \frac{t_1}{t_0}\,\mathrm{d}\left(\frac{t_0}{t_1}\right) = \mathrm{d}(1) = 0$. Using this relation we can simplify $d^0$ to

$$d^0\left(p\,\mathrm{d}\left(\frac{t_1}{t_0}\right), q\,\mathrm{d}\left(\frac{t_0}{t_1}\right)\right) = \left(q + \frac{t_1^2}{t_0^2}p\right) \mathrm{d}\left(\frac{t_0}{t_1}\right).$$

To calculate the kernel of $d^0$, we need to determine all $p, q$ with $\left(q + \frac{t_1^2}{t_2^2}p\right) = 0$. A Calculation similar to the one in the precessing example show that this is the case if and only if $p = q = 0$. Hence $\check{H}^0(\mathfrak{U}, \Omega_{\mathbb{P}^1_K/\operatorname{Spec} K}) = \ker d^0 = 0$.

It is easy to see that

$$\frac{K\left[\frac{t_0}{t_1}, \frac{t_1}{t_0}\right] \mathrm{d}\left(\frac{t_0}{t_1}\right)}{\operatorname{im} d^0} \cong K\,\mathrm{d}\left(\frac{t_0}{t_1}\right).$$

So $\check{H}^1(\mathfrak{U}, \Omega_{\mathbb{P}^1_K/\operatorname{Spec} K}) \cong K$. □

*Example* 3.20. Let $X = \mathbb{P}^1_K$ and let $\mathcal{F}$ be the constant sheaf given by $\mathbb{Z}$. Let $\mathfrak{U}$ again be $(U_0 = D_+(t_0), U_1 = D_+(t_1))$. Since $X$ is integral (and hence irreducible), $\mathcal{F}(U) = \mathbb{Z}$ for all nonempty open subsets $U$ of $X$. The Čech complex is

$$\begin{array}{c} 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d^0} \mathbb{Z} \longrightarrow 0 \\ (m, n) \longmapsto n - m \end{array}$$

Therefore we have

$$\begin{aligned} &\ker d^0 = \{(m, m) : m \in \mathbb{Z}\} \cong \mathbb{Z}, \\ &\operatorname{im} d^0 = \mathbb{Z}. \end{aligned}$$

Thus the cohomology groups are $\check{H}^0(\mathfrak{U}, \mathcal{F}) = \mathbb{Z}$ and $\check{H}^p(\mathfrak{U}, \mathcal{F}) = 0$ for $p \geq 1$. □

*Example* 3.21. Let $X$ be the circle $S^1$ with the usual topology induced by $\mathbb{R}^2$. Again let $\mathcal{F}$ be the constant sheaf defined by $\mathbb{Z}$. Chose to distinct points $a$ and $b$ on the circle and set $U = S^1 \setminus \{a\}$ and $V = S^1 \setminus \{b\}$. Cover $S^1$ with $\mathfrak{U} = (U, V)$. Since $U \cap V = S^1 \setminus \{a, b\}$ has two components, the Čech complex is

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{F}(U) \oplus \mathcal{F}(V) & \xrightarrow{d^0} & \mathcal{F}(U \cap V) & \longrightarrow 0 \\ & \| & & \| & \\ & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z}^2 & \\ & (m, n) & \longmapsto & (n - m, n - m) & \end{array}$$

Therefore $\check{H}^0(\mathfrak{U}, \mathcal{F}) \cong \mathbb{Z}$ and $\check{H}^1(\mathfrak{U}, \mathcal{F}) \cong \mathbb{Z}$. □

**Definition 3.22.** Let $X$ be a topological space. Let $\mathfrak{U} = (U_i)_{i \in I}$ be a finite open covering of $X$. Let $\mathcal{F}$ be a sheaf on $X$. Let $f_{i_0,\dots,i_p} \colon U_{i_0,\dots,i_p} \to X$ be the inclusion maps. Set

$$\mathcal{F}_{i_0,\dots,i_p} = \left(f_{i_0,\dots,i_p}\right)_* \mathcal{F}|_{U_{i_0,\dots,i_p}} \in \mathfrak{Sh}(X).$$

For $p \geq 0$ define

$$\mathcal{C}^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}_{i_0,\dots,i_p}.$$

Define $d^p \colon \mathcal{C}^p(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ analogous to definition 3.16. The complex of sheaves

$$0 \longrightarrow \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{d_0} \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{d_1} \mathcal{C}^2(\mathfrak{U}, \mathcal{F}) \xrightarrow{d_2} \cdots$$

is called *Čech complex.*

*Remark* 3.23. $\mathcal{C}(\mathfrak{U}, \mathcal{F})(X) = C(\mathfrak{U}, \mathcal{F})$.

**Lemma 3.24.** *With the notation of the definition, there exists a morphism $\mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F})$ such that the sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^1} \mathcal{C}^2(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^2} \cdots$$

*is exact.*

*Proof.* The map $\mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F})$ is defined by sending $s \in \mathcal{F}(W)$ to $(s|_{W \cap U_i})_i$ for each open subset $W$ of $X$. The exactness of

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F})$$

is just the sheaf condition for $\mathcal{F}$ as formulated in lemma 1.4.

To prove that the rest of the sequence is exact, we will show exactness on stalks. Pick any $x \in X$. We can assume that $x \in U_{1,\dots,n}$ where $I = \{1, \dots, n\}$ because if $x \notin U_j$ we can just ignore the index $j$ as it has no contribution to the stalk at $x$.

We will define maps

$$\mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x \xrightarrow{e_p} \mathcal{C}^{p-1}(\mathfrak{U}, \mathcal{F})_x.$$

for all $p \geq 1$: Let $(W, s) \in \mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x$, i.e. $x \in W$ and

$$s \in \mathcal{C}^p(\mathfrak{U}, \mathcal{F})(W) = \prod_{i_0 < \dots < i_p} \mathcal{F}(W \cap U_{i_0,\dots,i_p}).$$

Since $x \in U_{1,\dots,n}$, we can assume that $W \subseteq U_{1,\dots,n}$. Write $s = (s_{i_0,\dots,i_p})$. Define an element $t = (t_{i_0,\dots,t_{p-1}}) \in \mathcal{C}^{p-1}(\mathfrak{U}, \mathcal{F})_x$ by

$$t_{i_0,\dots,i_{p-1}} = \begin{cases} s_{1,i_0,\dots,i_{p-1}}, & \text{if } i_0 \neq 1 \\ 0, & \text{if } i_0 = 1 \end{cases},$$

where we assume that $I = \{1, 2, \dots\}$. Set $e_p(s) = t$. It turns out that on $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x$ we have the formula

$$e^{p+1} \circ d^p + d^{p-1} \circ e^p = \mathrm{id}\,.$$

$$\begin{tikzcd}
& \mathcal{C}^p_x \arrow[r, "d^p"] \arrow[dl, "e^p"'] \arrow[d, equal] & \mathcal{C}^{p+1}_x \arrow[dl, "e^{p+1}"] \\
\mathcal{C}^{p-1}_x \arrow[r, "d^{p-1}"] & \mathcal{C}^p_x &
\end{tikzcd}$$

Now if $d^p$ sends an element $(W, s) \in \mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x$ to zero in $\mathcal{C}^{p+1}(\mathfrak{U}, \mathcal{F})$, then $d^{p-1}e^p(W, s) = (W, s)$. Therefore $(W, s)$ is in the image of $d^{p-1}$. Thus the sequence is exact. $\square$

**Theorem 3.25.** *Let $X$ be a topological space, $\mathfrak{U}$ a finite open covering of $X$ and $\mathcal{F}$ a flasque sheaf on $X$. Then $\check{H}^p(\mathfrak{U}, \mathcal{F}) = 0$ for $p \geq 1$.*

*Proof.* Since $\mathcal{F}$ is flasque, all $\mathcal{F}|_{U_{i_0,\ldots,i_p}}$ are flasque which in turn implies that each $\mathcal{F}_{i_0,\ldots,i_p}$ is flasque. Therefore their direct sum $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ is flasque for all $p$. Thus lemma 3.24 provides a flasque resolution of $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^2(\mathfrak{U}, \mathcal{F}) \to \cdots \tag{3.3}$$

But cohomology can be calculated using flasque resolutions (corollary 3.10), so if we apply the global sections functor to the sequence (3.3)

$$0 \xrightarrow{d^{-1}} \mathcal{C}^0(\mathfrak{U}, \mathcal{F})(X) \xrightarrow{d^0} \mathcal{C}^1(\mathfrak{U}, \mathcal{F})(X) \xrightarrow{d^1} \mathcal{C}^2(\mathfrak{U}, \mathcal{F})(X) \xrightarrow{d^2} \cdots, \tag{3.4}$$

then $\frac{\ker d^p}{\operatorname{im} d^{p-1}} = H^p(X, \mathcal{F}) = 0$ for $p \geq 1$, since the homology groups for flasque sheaves vanish (theorem 3.9).

On the other hand the groups in the sequence (3.4) are just the groups $C^p(\mathfrak{U}, \mathcal{F})$ of the usual Čech complex, which calculate the Čech cohomology groups. So,

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = \frac{\ker d^p}{\operatorname{im} d^{p-1}} = H^p(X, \mathcal{F}) = 0 \qquad \text{for } p \geq 1. \qquad \square$$

**Theorem 3.26.** *Let $X$ be a separated (over $\operatorname{Spec} \mathbb{Z}$) Noetherian scheme, $\mathcal{F}$ a quasi-coherent sheaf on $X$ and $\mathfrak{U}$ a finite covering of $X$ by open affine subschemes. Then for all $p$,*

$$H^p(X, \mathcal{F}) = \check{H}^p(\mathfrak{U}, \mathcal{F}).$$

*Proof.* Let $\mathfrak{U} = (U_i)_{i \in I}$. For each $i \in I$ there exists an $\mathcal{O}_X(U_i)$-module $M_i$ such that $\mathcal{F}|_{U_i} = \widetilde{M_i}$. Let $J_i$ be an injective module such that $0 \to M_i \to J_i$ is exact. Then $\widetilde{J_i}$ is flasque and $0 \to \widetilde{M_i} \to \widetilde{J_i}$ remains exact. Let $\mathcal{G}_i$ be the direct image of $\widetilde{J_i}$ under the inclusion map $U_i \hookrightarrow X$. Set $\mathcal{G} = \prod \mathcal{G}_i$. Then $\mathcal{G}$ is a flasque, quasi-coherent sheaf on $X$ and $\mathcal{F}$ injects into $\mathcal{G}$. Let $\mathcal{H} = \mathcal{G}/\mathcal{F}$. Then $\mathcal{H}$ is quasi-coherent and there is a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.$$

Consider any sequence $i_0 < i_1 < \cdots < i_p$. Since $X$ is separated, $U_{i_0,\ldots,i_p}$ is affine (exercise A.24). The long exact sequence of cohomology gives

$$0 \to \mathcal{F}(U_{i_0,\ldots,i_p}) \to \mathcal{G}(U_{i_0,\ldots,i_p}) \to \mathcal{H}(U_{i_0,\ldots,i_p}) \to \underbrace{H^1(U_{i_0,\ldots,i_p}, \mathcal{F}|_{U_{i_0,\ldots,i_p}})}_{=0 \text{ by theorem 3.15}}$$

The direct sum of all these sequences gives an exact sequence

$$0 \to C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{U}, \mathcal{G}) \to C^\bullet(\mathfrak{U}, \mathcal{H}) \to 0$$

By theorem 1.37 we get a long exact sequence.

$$\cdots \to \check{H}^p(\mathfrak{U}, \mathcal{F}) \to \check{H}^p(\mathfrak{U}, \mathcal{G}) \to \check{H}^p(\mathfrak{U}, \mathcal{H}) \to \check{H}^{p+1}(\mathfrak{U}, \mathcal{F}) \to \cdots$$

Since $\mathcal{G}$ is flasque, $\check{H}^p(\mathfrak{U}, \mathcal{G}) = 0 = H^p(X, \mathcal{G})$ for all $p \geq 1$ (theorems 3.9 and 3.25). In particular, we have

$$\check{H}^0(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathcal{G}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathcal{H}) \longrightarrow \check{H}^1(\mathfrak{U}, \mathcal{F}) \longrightarrow 0$$

$$H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow 0$$

We already know that $\check{H}^0(\mathfrak{U}, \mathcal{F}) = H^0(X, \mathcal{F}) = \mathcal{F}(X)$ (Theorem 3.17; the same is true for $\mathcal{G}$ and $\mathcal{H}$). The maps on the 0-th cohomology groups are just the maps on global sections. So the cokernels are the same, i.e. $\check{H}^1(\mathfrak{U}, \mathcal{F}) \cong H^1(X, \mathcal{F})$ and the analogous statement holds for $\mathcal{H}$. Parts of the long exact sequence look like

$$0 \longrightarrow \check{H}^p(\mathfrak{U}, \mathcal{H}) \longrightarrow \check{H}^{p+1}(\mathfrak{U}, \mathcal{F}) \longrightarrow 0$$

$$0 \longrightarrow H^p(X, \mathcal{H}) \longrightarrow H^{p+1}(X, \mathcal{F}) \longrightarrow 0$$

This allows us to do induction to show the result for all $p$. □

## 3.3. Projective Space

**Lemma* 3.27.** *Let $\pi \colon Y \to X$ be a closed subscheme. Let $\mathcal{F}$ be a sheaf on $Y$. Then $H^i(Y, \mathcal{F}) = H^i(X, \pi_* \mathcal{F})$.* [Har77, lem. III.2.10]

*Proof.* See exercise A.34. □

**Theorem 3.28.** *Let $k$ be a field. Let $X = \mathbb{P}^n_k = \operatorname{Proj} k[t_0, \ldots, t_n]$. Let $d \in \mathbb{Z}$. Then:*

1. *$H^0(X, \mathcal{O}_X(d))$ is isomorphic to the $k$-vector space generated by monomials of degree $d$ in $t_0, \ldots, t_n$. In particular, if $d < 0$, then it is $0$.*
2. *$H^n(X, \mathcal{O}_X(d)) \cong H^0(X, \mathcal{O}_X(-d-n-1))$.*
3. *$H^p(X, \mathcal{O}_X(d)) = 0$ if $0 < p < n$ or $p > n$.*

*Proof.* Note that $X$ is Noetherian and separated over $\operatorname{Spec} k$ (and hence over $\mathbb{Z}$). Put $\mathfrak{U} = (U_i)_{i=0,\ldots,n}$ with $U_i = D_+(t_i)$, so that we can apply theorem 3.26 and calculate cohomology via the Čech complex. Set $S = k[t_0, \ldots, t_n]$.

1. We always have $H^0(X, \mathcal{O}_X(d)) = \mathcal{O}_X(d)(X)$. Every element of $s$ of $\mathcal{O}_X(d)(X)$ is uniquely determined by a system $(s_i)$ with $s_i \in \mathcal{O}_X(d)(U_i) = S(d)_{(t_i)}$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Each $s_i$ can be written in the form $f_i/t_i^{l_i}$ where $f_i$ is a homogeneous polynomial with $\deg f_i = l_i + d$. The condition $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ translates to $f_i/t_i^{l_i} = f_j/t_j^{l_j}$ in $k(t_0, \ldots, t_n)$. In particular, $f_i t_j^{l_j} = f_j t_i^{l_i}$ in $S$ and hence $t_i^{l^i}$ divides $f_i$ and $t_j^{l_j}$ divides $f_j$. Thus $f_i/t_i^{l_i}$ is in $S$, is independent of $i$ and has degree $d$. So $\mathcal{O}_X(d)(X)$ is generated as a $k$-vector space by the homogeneous polynomials of degree $d$.
2. The Čech complex ends with

$$\cdots \longrightarrow C^{n-1}(\mathfrak{U}, \mathcal{O}_X(d)) \xrightarrow{d^{n-1}} \underbrace{C^n(\mathfrak{U}, \mathcal{O}_X(d))}_{\substack{=\mathcal{O}_X(d)(U_0 \cap \cdots \cap U_n) \\ =S(d)_{(t_0 \cdots t_n)}}} \longrightarrow 0$$

Every element of $C^n(\mathfrak{U}, \mathcal{O}_X(d))$ is of the form $\alpha = \frac{f}{(t_0 \cdots t_n)^l}$ such that $\deg f = (n+1)l + d$ (in $S$), where we choose $l$ such that it is minimal. Assume that $f$ is a monomial, say $f = t_0^{m_0} \cdots t_n^{m_n}$ If $m_i \geq l$ for some $i$, then $\alpha = \frac{t_0^{m_0} \cdots t_i^{m_i - l} \cdots t_n^{m_n}}{(t_0 \cdots \widehat{t_i} \cdots t_n)^l} \in \operatorname{im} d^{n-1}$. So assume that all $m_i < l$. Also, at least one $m_i$ is zero as otherwise $l$ would not be minimal. Then $(n+1)l + d = \deg f \leq (l-1)n$, so $d \leq -n - l \leq -n - 1$. So, if $d > -n - 1$, then $f$ is always in $\operatorname{im} d^{n-1}$ and $H^n(X, \mathcal{O}_X(d)) = 0 = H^0(X, \mathcal{O}_X(-d-n-1))$.

Let $d = -n-1$. Then $l = 1$ and $\deg f = 0$. Thus $H^n(X, \mathcal{O}_X(d)) = C^n(\mathfrak{U}, \mathcal{O}_X(d))/\operatorname{im} d^{n-1}$ is generated by (the coset of) the single element $\frac{1}{t_0 \cdots t_n}$. Hence $H^n(X, \mathcal{O}_X(-n-1)) \cong k \cong H^0(X, \mathcal{O}_X)$.

We will not prove (2) for $d < -n-1$.

3. Note that we will implicitly use lemma 3.27 wherever needed. From the Čech complex we immediately see that the cohomology vanishes for $p > n$. In particular, the statement is trivial for $n = 1$. So assume that $n \geq 2$.

   Let $Y$ be the closed subscheme of $X$ defined by $t_n = 0$. Then $Y \cong \mathbb{P}^{n-1}_k$. Via the homomorphism $f \mapsto ft_n$, we see that $S(-1) \cong (t_n)$ (as graded $S$-modules). So there is an exact sequence

$$0 \to S(-1) \xrightarrow{\cdot t_n} S \to S/(t_n) \to 0.$$

   Since $\widetilde{S(-1)} = \mathcal{O}(-1)$ ($\cong \mathcal{I}_Y$), $\widetilde{S} = \mathcal{O}_X$ and $\widetilde{S/(t_n)} \cong \widetilde{k[t_0, \ldots, t_{n-1}]} \cong \pi_* \mathcal{O}_Y$ (where $\pi \colon Y \to X$ is the closed immersion), the sequence of $S$-modules gives the following exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \pi_* \mathcal{O}_Y \to 0.$$

   If we tensor the sequence with $\mathcal{O}_X(d)$ we get the exact sequence

$$0 \to \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \to \pi_* \mathcal{O}_Y(d) \to 0.$$

   This gives a long exact sequence on cohomology:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{O}_X(d-1)) \arrow[r] & H^0(X, \mathcal{O}_X(d)) \arrow[r] & H^0(X, \pi_* \mathcal{O}_Y(d)) \arrow[dll, "\alpha^0"] & \\
& H^1(X, \mathcal{O}_X(d-1)) \arrow[r] & H^1(X, \mathcal{O}_X(d)) \arrow[r] & H^1(X, \pi_* \mathcal{O}_Y(d)) \arrow[dll, dashed] & \\
& H^{n-1}(X, \mathcal{O}_X(d-1)) \arrow[r] & H^{n-1}(X, \mathcal{O}_X(d)) \arrow[r] & H^{n-1}(X, \pi_* \mathcal{O}_Y(d)) \arrow[dll, "\alpha^{n-1}"] & \\
& H^n(X, \mathcal{O}_X(d-1)) \arrow[r] & H^n(X, \mathcal{O}_X(d)) \arrow[r] & \underbrace{H^n(X, \pi_* \mathcal{O}_Y(d))}_{=0} \arrow[r] & 0
\end{tikzcd}$$

   Using parts (1) and (2) to count the dimensions of the $k$-vector spaces $H^0(X, \mathcal{O}_X(d-1))$, $H^0(X, \mathcal{O}_X(d))$, $H^0(X, \pi_* \mathcal{O}_Y(d))$ and $H^{n-1}(X, \pi_* \mathcal{O}_Y(d))$, $H^n(X, \mathcal{O}_X(d-1))$, $H^n(X, \mathcal{O}_X(d))$, we see that $\alpha^0$ is the zero map and $\alpha^{n-1}$ is an injection.

   By induction on $n$, we know that $H^p(X, \pi_* \mathcal{O}_Y(d)) = 0$ for $0 < p < n-1$. Together with our knowledge about $\alpha^0$ and $\alpha^{n-1}$, this implies that the maps

$$H^p(X, \mathcal{O}_X(d-1)) \xrightarrow{\beta^p} H^p(X, \mathcal{O}_X(d))$$

   are isomorphisms for $0 < p < n$. The maps $\beta^p$ come from the maps

$$\begin{tikzcd}
\mathcal{O}_X(d-1)(U_{i_0,\ldots,i_p}) \arrow[r] \arrow[d, equal] & \mathcal{O}_X(d)(U_{i_0,\ldots,i_p}) \arrow[d, equal] \\
S(d-1)_{(t_{i_0} \cdots t_{i_p})} \arrow[r, "\cdot t_n"] & S(d)_{(t_{i_0} \cdots t_{i_p})}
\end{tikzcd}$$

   (Recall that $S(-1) \to S$ is given by multiplication with $t_n$.) So $\beta^p$ is given by multiplication with $t_n$.

We will prove that if $\omega \in H^p(X, \mathcal{O}_X(d))$, then $t_n^l \cdot \omega = 0$ for some $l$. This implies that $\beta^p$ is the zero map so that $H^p(X, \mathcal{O}_X(d)) = 0$ for $0 < p < n$.

Set

$$\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(d) = \widetilde{\bigoplus_{d \in \mathbb{Z}} S(d)}.$$

Now it turns out that

$$\mathcal{F}(U_{i_0,\ldots,i_p}) = \left( \bigoplus_{d \in \mathbb{Z}} S(d) \right)_{(t_{i_0} \cdots t_{i_p})} \cong S_{t_{i_0} \cdots t_{i_p}}.$$

Thus the Čech complex for $\mathcal{F}$ looks like

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^0(\mathfrak{U}, \mathcal{F}) & \longrightarrow & C^1(\mathfrak{U}, \mathcal{F}) & \longrightarrow & \cdots \\ & & \| & & \| & & \\ & & \prod_{i_0} S_{t_{i_0}} & & \prod_{i_0, i_1} S_{t_{i_0} t_{i_1}} & & \end{array}$$

If we localize this sequence at $t_n$, we get

$$0 \to \prod S_{t_{i_0} t_n} \to \prod S_{t_{i_0} t_{i_1} t_n} \to \cdots$$

This is the same as the Čech complex of $\mathcal{F}|_{U_n}$ given by the covering $U_0 \cap U_n, \ldots, U_n \cap U_n$. Now $H^p(U_n, \mathcal{F}|_{U_n}) = 0$ for $p \geq 1$ because $U_n$ is affine (theorem 3.15). Moreover $H^p(X, \mathcal{F})_{t_n} = H^p(U_n, \mathcal{F}|_{U_n}) = 0$ so that for any $\omega \in H^p(X, \mathcal{F})$ we have $t_n^l \omega = 0$ for $l$ sufficiently large. □

## 3.4. The Riemann-Roch Theorem

Let $k$ be a fixed algebraically closed field.

**Definition 3.29.** An integral quasi-projective scheme $X$ over $k$ of dimension $d$ is *smooth* if for every closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a regular local ring of dimension $d$.

*Remark* 3.30. A smooth integral quasi-projective scheme over $k$ is Noetherian.

**Definition 3.31.** Let $X$ be a smooth integral quasi-projective scheme over $k$ of dimension $d$. A *Weil divisor* on $X$ is a formal sum $\sum m_i M_i$ where $m_i \in \mathbb{Z}$ and $M_i = \overline{\{\mu_i\}}$ with $\mu_i \in X$ such that $\mathcal{O}_{X,\mu_i}$ is of dimension one (i.e. a DVR) and all but finitely many $m_i$ are 0.

Let $f \in K(X)^*$. The Weil divisor of $f$ is

$$(f) = \sum_{\substack{\mu \in X \\ \mathcal{O}_{x,\mu} \text{ of dim } 1}} v_\mu(f) \bar{\mu},$$

where $v_\mu$ is the normalized valuation $K(X)^* \to \mathbb{Z}$ of the DVR $\mathcal{O}_{X,\mu}$.

*Remark* 3.32. The divisor $(f)$ is indeed a finite sum: Since $X$ is Noetherian it is enough to check the finiteness on open affine subschemes and then cover $X$ with finitely many of them. So let $U = \operatorname{Spec} A$ be an affine subscheme of $X$. Then $K(X)$ is just the fraction field of $A$ and hence we can write $f = \frac{a}{b}$ with $a, b \in A$. If $v_\mu(f) \neq 0$ for some $\mu \in U$, then $\mu \in V((a)) \cup V((b)) = V((ab))$. There are only finitely many such $\mu$ in $V((ab))$ (as in Noetherian rings there are only finitely many minimal primes over any ideal [?]), and hence only finitely many in $U$.

**Definition 3.33.** Let $X$ be a smooth integral projective scheme over $k$ of dimension 1. Then the *degree* of a Weil divisor $M = \sum m_i M_i$ is $\deg M = \sum m_i \in \mathbb{Z}$.

**Fact 3.34.** *Let $X$ be as in the definition and $f \in K(X)^*$. Then* $\deg(f) = 0$.

**Theorem 3.35.** *Let $X$ be a smooth integral projective scheme over $k$ of dimension 1. Then there is a one-to-one correspondence between Cartier and Weil divisors.*

*Proof.* Suppose that $D$ is a Cartier divisor given by the system $(U_i, f_i)$. On $U_i$ consider the Weil divisor $(f_i)$. We can view $(f_i)$ as a Weil divisor on $X$ by taking the closure of each component. On any $U_i \cap U_j$ we have $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i}$ both in $\mathcal{O}_X^*(U_i \cap U_j)$. This means that $(f_i)$ and $(f_j)$ coincide on $U_i \cap U_j$, so that the whole system can be put together to give a single Weil divisor on $X$.

Conversely let $M = \sum m_\alpha M_\alpha$ (with $M_\alpha$ closed points) be a Weil divisor on $X$. Then define a Cartier divisor as follows: Pick a closed point $x \in X$. The multiplicity of $x$ in $M$ is just $m_\alpha$ if $M_\alpha = x$ (or 0 if $x$ does not appear in $M$). Since $\mathcal{O}_x$ is a DVR, the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_x$ is generated by a single element, say $t$. Set $f = t^{m_\alpha}$. Then $M = (f)$ in some open affine neighborhood of $x$ (in any affine neighborhood there are only finitely many points where this could be false (Remark 3.32) and we can simply remove those points from the neighborhood). Now $X$ is Noetherian, so it can be covered by finitely many such open affine schemes. This means that we get a system $(U_i, f_i)$ such that $U_i$ cover $X$ and $(f_i) = M$ on $U_i$. Since $(f_i) = (f_j)$ on $U_i \cap U_j$, $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i}$ are in $\mathcal{O}_X^*(U_i \cap U_j)$. So $(U_i, f_i)$ defines a Cartier divisor. □

**Definition 3.36.** Let $X$ be a smooth integral projective scheme over $k$ of dimension 1. Let $D$ be a Cartier divisor on $X$. Then the *degree* of $D$, denoted $\deg D$, is the degree of the corresponding Weil divisor.

*Remark* 3.37. In the setting of the definition, if $D \sim D'$, then $\deg D = \deg D'$.

**Theorem 3.38.** *Let $X$ be a smooth integral projective scheme over $k$. Let $D$ be a Cartier divisor on $X$. Then all cohomology groups $H^p(X, \mathcal{O}_X(D))$ are finite dimensional $k$-vector spaces.*

*Proof.* Not given. □

**Theorem 3.39** (Duality)**.** *Let $X$ be a smooth integral projective scheme over $k$ of dimension $d$. Then there exists a Cartier divisor $K_X$ such that for every Cartier divisor $D$ on $X$,*

$$\dim_k H^p(X, \mathcal{O}_X(D)) = \dim_k H^{d-p}(X, \mathcal{O}_X(K_X - D)).$$

*Proof.* Not given. □

[Har77, IV.1.3] **Theorem 3.40** (Riemann-Roch)**.** *Let $X$ be a smooth integral projective scheme over $k$ of dimension 1. Let $D$ be any Cartier divisor on $X$. Then*

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D)) = \deg D + 1 - \dim_k H^0(X, \mathcal{O}_X(K_X)).$$

*Proof.* Let $M = \sum m_i M_i$ be the Weil divisor corresponding to $D$. Let $x \in X$ be a closed point. Write $D_x$ for the Cartier divisor associated to Weil divisor $x$. Let $D$ be given by a system $(U_i, f_i)$ where $U_i$ are open affine and let $(U_i, g_i)$ be a system for $D_x$. If $W \subseteq X$ is any open subscheme, then

$$\begin{array}{c} \mathcal{O}_X(D)(W) = \\ \{h \in K(X) : hf_i \in \mathcal{O}_X(U_i \cap W)\ \forall i\} \end{array} \xrightarrow{h \mapsto h} \begin{array}{c} \mathcal{O}_X(D + D_x)(W) = \\ \{e \in K(X) : ef_i g_i \in \mathcal{O}_X(U_i \cap W)\ \forall i\} \end{array}$$

is a well-defined homomorphism: since the Weil divisor of $D_x$ is just $x$, all $g_i$ are in $\mathcal{O}_X(U_i)$; so if $hf_i \in \mathcal{O}_X(W \cap U_i)$, then $hf_i g_i \in \mathcal{O}_X(W \cap U_i)$. These maps give an injective morphism $\mathcal{O}_X(D) \to \mathcal{O}_X(D + D_x)$.

The quotient sheaf $\mathcal{O}_X(D+D_x)/\mathcal{O}_X(D)$ is the skyscraper sheaf $k(x)$ at $x$ defined by $k$: if $W \subseteq X$ is an open affine and $x \notin W$, then $g_i \in \mathcal{O}_X^*(W \cap U_i)$. Thus there is an exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+D_x) \to k(x) \to 0.$$

The corresponding long exact sequence of cohomology is

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{O}_X(D)) \arrow[r] & H^0(X,\mathcal{O}_X(D+D_x)) \arrow[r] & H^0(X,k(x)) \arrow[dll, out=-10, in=170] \\ & H^1(X,\mathcal{O}_X(D)) \arrow[r] & H^1(X,\mathcal{O}_X(D+D_x)) \arrow[r] & H^1(X,k(x)) \arrow[r] & 0 \end{tikzcd}$$

Now $H^1(X,k(x))=0$. So, by duality, we obtain the following sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{O}_X(D)) \arrow[r] & H^0(X,\mathcal{O}_X(D+D_x)) \arrow[r] & H^0(X,k(x)) \arrow[dll, out=-10, in=170] \\ & H^0(X,\mathcal{O}_X(K_X-D)) \arrow[r] & H^0(X,\mathcal{O}_X(K_X-D-D_x)) \arrow[r] & 0 \end{tikzcd}$$

Using linear algebra we obtain the formula

$$\begin{aligned} &\dim_k H^0(X,\mathcal{O}_X(D+D_x)) - \dim_k H^0(X,\mathcal{O}_X(K_X-D-D_x)) = \\ &\qquad \dim_k H^0(X,\mathcal{O}_X(D)) - \dim_k H^0(X,\mathcal{O}_X(K_x-D)) + \underbrace{\dim_k H^0(X,k(x))}_{=1}. \end{aligned}$$

Therefore the theorem holds for a divisor $D$ if and only if it holds for $D+D_x$ for any closed point $x$ (note that $\deg(D+D_x)=\deg D+1$). So by removing and adding points to $M$, we can reduce to the case $M=0$. But in this case $\dim_k H^0(X,\mathcal{O}_X(D))=1$ [exercise; hint: prove that $H^0(X,\mathcal{O}_X(D))=H^0(X,\mathcal{O}_X)$ and that $\mathcal{O}_X(X)$ is a finitely generated $k$-algebra which is a field] and $\deg D=0$, so that the theorem holds. $\square$

# A. Example Sheets

## A.1. Sheet 1

*Exercise* A.1. Give an example of a topological space $X$ and a surjective morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ of sheaves, such that $\mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective.

*Solution.* Consider the three-point-space $X = \{a, b, c\}$ with the topology given by the open sets

$$X,\ U = \{a, b\},\ V = \{b, c\},\ U \cap V = \{b\},\ \emptyset.$$

On $X$ consider the sheaves $\mathcal{F}$, $\mathcal{G}$ and the map $\mathcal{F} \to \mathcal{G}$ as indicated in the following diagram (where $A$ is any nontrivial Abelian group):

$$\begin{tikzcd}
 & \mathcal{F}(U) = A \arrow[dr, "\mathrm{id}"] \arrow[ddd, dashed, bend right, "\mathrm{id}"'] & & \\
\mathcal{F}(X) = A \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] \arrow[ddd, dashed, "a \mapsto (a,a)"'] & & \mathcal{F}(U \cap V) = A \arrow[r] \arrow[ddd, dashed] & \mathcal{F}(\emptyset) = 0 \arrow[ddd, dashed] \\
 & \mathcal{F}(V) = A \arrow[ur, "\mathrm{id}"'] \arrow[ddd, dashed, bend left, "\mathrm{id}"] & & \\
 & \mathcal{G}(U) = A \arrow[dr] & & \\
\mathcal{G}(X) = A \times A \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & \mathcal{G}(U \cap V) = 0 \arrow[r] & \mathcal{G}(\emptyset) = 0 \\
 & \mathcal{G}(V) = A \arrow[ur] & &
\end{tikzcd}$$

where $p_1$ and $p_2$ are the projections on the first resp. second coordinate. The map $\mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective, but the morphism $\mathcal{F} \to \mathcal{G}$ is because the maps on the stalks are

$$\mathcal{F}_a = A \xrightarrow{\mathrm{id}} \mathcal{G}_a = A$$
$$\mathcal{F}_b = A \longrightarrow \mathcal{G}_b = 0$$
$$\mathcal{F}_c = A \xrightarrow{\mathrm{id}} \mathcal{G}_c = A$$

A different (maybe more natural) counterexample is the following: Let $X = \mathbb{C} \setminus \{0\}$ and consider as sheaves $\mathcal{F}$ and $\mathcal{G}$ both the sheaf of non-vanishing holomorphic functions. Let the map $\phi\colon \mathcal{F} \to \mathcal{G}$ be defined by $\phi_U : f \mapsto f^2$ on each open subset $U$. By complex analysis (e.g. [Con78, Theorem VIII.2.2]): An open connected subset $G$ of $\mathbb{C}$ is simply connected if and only if for all holomorphic functions $g$ on $G$ there exists a holomorphic function $f$ on $G$ such that $f^2 = g$. Since $X$ is has a simply connected topological base, this implies that $\phi$ is surjective. On the other hand $X$ itself is not simply connected. Hence $\phi_X : \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective. $\square$

*Exercise* A.2. Let $\mathcal{F}$ be a sheaf on a topological space $X$, and let $U \subseteq X$ be an open subset. For each open subset $V \subseteq U$ put $\mathcal{F}|_U(V) = \mathcal{F}(V)$. Show that $\mathcal{F}|_U$ is a sheaf on $U$ which is called the *restriction* of $\mathcal{F}$ to $U$.

*Solution.* TBA $\square$

*Exercise* A.3. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves on a topological space $X$. For any open set $U \subseteq X$ let

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) = \{\text{morphisms } \phi\colon \mathcal{F}|_U \to \mathcal{G}|_U\}.$$

Show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf on $X$.

*Solution.* TBA □

*Exercise* A.4. Let $X$ be a topological space and let $X = \bigcup U_i$ be an open cover. Suppose that for each $i$ we have a sheaf $\mathcal{F}_i$ in $U_i$. Moreover, assume that for each pair $i$, $j$ we have an isomorphism $\phi_{ij}\colon \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ such that: for each $i$, $\phi_{ii}$ is the identity morphism, and for each $i$, $j$, $k$ we have $\phi_{ik} = \phi_{jk}\phi_{ij}$ on $U_i \cap U_j \cap U_k$. Show that there exists a unique sheaf $\mathcal{F}$ on $X$ and isomorphisms $\psi_i\colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that for each pair $i$, $j$ we have $\psi_j = \phi_{ij}\psi_i$ on $U_i \cap U_j$. The sheaf $\mathcal{F}$ is said to be the sheaf obtained by *gluing* the sheaves $\mathcal{F}_i$.

*Solution.* TBA □

*Exercise* A.5. Let $X$ be a topological space, and $\mathcal{F}$, $\mathcal{G}$ sheaves on $X$. For any open subset $U \subseteq X$ let $\mathcal{F} \oplus \mathcal{G}(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$. Show that $\mathcal{F} \oplus \mathcal{G}$ is a sheaf on $X$ which is called the *direct sum* of $\mathcal{F}$, $\mathcal{G}$.

*Solution.* TBA □

*Exercise* A.6. A topological space $X$ is said to be Noetherian if for any sequence $U_1 \subseteq U_2 \subseteq \cdots$ of open subsets, there is $n$ such that $U_i = U_{i+1}$ for any $i \geq n$. Now assume that $X$ is a Noetherian topological space. Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on $X$, and for any open subset $U \subseteq X$ let $(\varinjlim \mathcal{F}_i)(U) = \varinjlim(\mathcal{F}_i(U))$. Show that $\varinjlim \mathcal{F}_i$ is a sheaf. Prove the same statement replacing direct system with inverse system and replacing direct limit with inverse limit (in this case you do not need the Noetherian property).

*Solution.* TBA □

*Exercise* A.7. Give an example of a locally ringed space which is not of the form $(X = \operatorname{Spec} A, \mathcal{O}_X)$ for any ring $A$.

*Solution.* TBA □

*Exercise* A.8. Let $A$ be a ring and $X$ a scheme. Show that there exists a 1-1 correspondence between the set of ring homomorphisms $A \to \mathcal{O}_X(X)$ and the set of morphisms of schemes $X \to \operatorname{Spec} A$. Determine these sets when $A = \mathbb{Z}$, the ring of integers.

*Solution.* Any morphism $X \to \operatorname{Spec} A$ induces a homomorphism on global sections $A \to \mathcal{O}_X(X)$.

Conversely, suppose we are given a homomorphism $A \to \mathcal{O}_X(X)$. For any open affine $\operatorname{Spec} B = U \subseteq X$ the restriction homomorphism $\mathcal{O}_X(X) \to \mathcal{O}_X(U) = B$ induces a homomorphism $A \to B$. This in turn induces a morphism of schemes $U \to \operatorname{Spec} A$. We have to check that we can glue all the morphisms together. Let $V = \operatorname{Spec} C$ be another affine open subset of $X$ and $W = \operatorname{Spec} D \subseteq U \cap V$. Then we have a commutative diagram of rings

$$\begin{tikzcd}
 & & B \arrow[dr] & \\
A \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \mathcal{O}_X(X) \arrow[ur] \arrow[dr] & & D \\
 & & C \arrow[ur] &
\end{tikzcd}$$

which induces a commutative diagram of schemes

$$\begin{tikzcd} & \operatorname{Spec} B \arrow[dl] & \\ \operatorname{Spec} A & & \operatorname{Spec} D \arrow[ll] \arrow[ul] \arrow[dl] \\ & \operatorname{Spec} C \arrow[ul] & \end{tikzcd}$$

□

*Exercise* A.9. Let $X$ be a scheme, and $L$ a field. Show that to give a morphism $\operatorname{Spec} L \to X$ is the same as giving a point $x \in X$ and a field extension $k(x) := \frac{\mathcal{O}_x}{\mathfrak{m}_x} \to L$ where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_x$. The field $k(x)$ is called the *residue field* of $x$ on $X$.

*Solution.* TBA □

*Exercise* A.10. Let $X$ be a scheme and $x \in X$. Show that there is a natural morphism $\operatorname{Spec} \mathcal{O}_x \to X$.

*Solution.* TBA □

*Exercise* A.11. Let $\mathbb{F}_p$ be the finite field with $p$ elements. Describe $\operatorname{Spec} \mathbb{F}_p[t]$. What are the residue fields of its points?

*Solution.* TBA □

*Exercise* A.12. Let $X$ be a scheme. The *Zariski tangent space* $T_x$ at $x$ is defined to be the dual of the $k(x)$-vector space $\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$. Assume that $X$ is defined over a field $L$, i.e., there is a given morphism $f\colon X \to \operatorname{Spec} L$. Show that to give a morphism $\operatorname{Spec} L[t]/(t^2) \to X$ over $L$, i.e., to give a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \frac{L[t]}{t^2} \arrow[r] \arrow[dr] & X \arrow[d, "f"] \\ & \operatorname{Spec} L \end{tikzcd}$$

is the same as giving a point $x \in X$ with $k(x) = L$ and an element of $T_x$.

*Solution.* We will write $Y = \operatorname{Spec}\big(L[t]/(t^2)\big)$. Note that $Y$ consists of the single point $y = (t)$.

Suppose we are given a morphism $g\colon Y \to X$ of $L$-schemes. Set $x = g(y)$. Then $f$ and $g$ induce inclusions $L \subseteq k(x) \subseteq k(y) = L$ and so $k(x) = L$. The local homomorphism $\mathcal{O}_x \to \mathcal{O}_y$ restricts to an $L$-homomorphism $\alpha\colon \mathfrak{m}_x \to \mathfrak{m}_y$. The ideal $\mathfrak{m}_y = (t) \subseteq L[t]/(t^2)$ is isomorphic as an $L$-vector space to $L$ (via $at \mapsto a$). Also $\mathfrak{m}_y^2 = 0$, so that $\mathfrak{m}_x^2 \subseteq \ker \alpha$. Thus we get an $L$-linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to L$, i.e. an element in $T_x$.

Conversely, suppose we are given $x \in X$ with $k(x) = L$ and an $L$-linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to L$. From the latter we obtain a map $\alpha\colon \mathfrak{m}_x \to L$. Let $\beta\colon \mathcal{O}_x \to \mathcal{O}_x/\mathfrak{m}_x = L$ be the projection map. The composition $L \to \mathcal{O}_x \xrightarrow{\beta} L$ is the identity. So for any element $a \in \mathcal{O}_x$, $\beta(a)$ is the unique element such that $a - \beta(a) \in \mathfrak{m}_x$. Now define a map

$$\gamma\colon \mathcal{O}_x \to L[t]/(t^2), \qquad a \mapsto \beta(a) + \alpha(a - \beta(a))t.$$

Then $\gamma$ is clearly a homomorphism of Abelian groups and is $L$-linear. To check that is a ring homomorphism we compute

$$\gamma(a)\gamma(b) = (\beta(a) - \alpha(a - \beta(a))t)\,(\beta(b) - \alpha(b - \beta(b))t) = \\ \beta(a)\beta(b) + \alpha(\beta(a)b + a\beta(b) - 2\beta(a)\beta(b))t = \beta(ab) + \alpha(ab - \beta(ab)) = \gamma(ab)$$

since

$$ab - \beta(a)b - a\beta(b) + \beta(a)\beta(b) = (a - \beta(a))(b - \beta(b)) \in \mathfrak{m}_x^2.$$

Hence $\gamma$ induces an $L$-morphism $Y \to X$.

The two constructions are inverses. □

*Exercise* A.13. Let $\alpha\colon A \to B$ be a homomorphism of rings and $f\colon Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ the induced morphism of schemes.

1. Show that $\alpha$ is injective if and only if the morphism $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective.
2. Show that $\alpha$ is surjective if and only if $f$ is a homeomorphism onto a closed subset of $X$ and $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

*Solution.* TBA □

*Exercise* A.14. Consider a homomorphism $\alpha\colon S \to T$ of graded rings which preserves degrees, and let $U = \{P \in \operatorname{Proj} T : \alpha^{-1}P \in \operatorname{Proj} S\}$. Show that we naturally get an induces morphism $f\colon U \to \operatorname{Proj} S$.

Find an example of $\alpha$ which is not a isomorphism but such that the induced morphism $f$ is an isomorphism.

*Solution.* TBA □

*Exercise* A.15. Let $A$ be a ring. Prove that the following are equivalent:

1. $\operatorname{Spec} A$ is not connected as a topological space.
2. There are nonzero element $e, e' \in A$ such that $ee' = 0$, $e^2 = e$, $e'^2 = e'$, and $e + e' = 1$.
3. There are nonzero rings $A'$ and $A''$ such that $A$ is isomorphic to the product $A \times A'$.

*Solution.* TBA □

*Exercise* A.16. Describe $\operatorname{Spec} \mathbb{Z}[t]$ and the morphism $\operatorname{Spec} \mathbb{Z}[t] \to \operatorname{Spec} \mathbb{Z}$ induced by the homomorphism $\mathbb{Z} \to \mathbb{Z}[t]$.

*Solution.* TBA □

*Exercise* A.17. Describe $\operatorname{Spec} \mathbb{R}[t_1, t_2]$ and the morphism $\operatorname{Spec} \mathbb{C}[t_1, t_2] \to \operatorname{Spec} \mathbb{R}[t_1, t_2]$ induced by the homomorphism $\mathbb{R} \to \mathbb{C}$.

*Solution.* TBA □

*Exercise* A.18. Formulate and prove a statement similar to A.4 for gluing schemes (see [Har77, exercise II.2.12]).

*Solution.* TBA □

*Exercise* A.19. Let $\{A_i\}$ be a direct system of rings and let $A = \varinjlim A_i$. Show that the inverse limit $\varprojlim \operatorname{Spec} A_i$ exists in the category of affine schemes and it is isomorphic to $\operatorname{Spec} A$.

*Solution.* TBA □

## A.2. Sheet 2

*Exercise* A.20. Let $X$ be an integral scheme. Show that $X$ has a generic point $\eta$, i.e. $\eta$ is dense in $X$. Show that the local ring $\mathcal{O}_\eta$ is a field which is called the *function field* of $X$ denoted by $K(X)$. Moreover show that this field is the fraction field of $\mathcal{O}_X(U)$ for any open affine subscheme $U \subseteq X$.

*Solution.* TBA □

*Exercise* A.21. Show that every closed subscheme of $X = \operatorname{Spec} A$ is uniquely determined by an ideal $\mathfrak{a}$ of $A$ (we show this in corollary 2.87, however, for an elementary proof you might like to consult [Har77, exercise II.3.11]).

*Solution.* TBA □

*Exercise* A.22. Let $\alpha\colon S \to T$ be a surjective graded morphism of graded rings. Show that this induces a morphism $\operatorname{Proj} T \to \operatorname{Proj} S$ which is a closed immersion.

*Solution.* TBA □

*Exercise* A.23. Separatedness is local on the base, i.e. a morphism $f\colon X \to Y$ of Noetherian schemes is separated if and only if $Y$ can be covered by open subschemes $U_i$ such that the induced morphisms $f^{-1}U_i \to U_i$ are separated.

*Solution.* TBA □

Note that the same is true for properness (with an analogous proof).

*Exercise* A.24. Let $f\colon X \to Y$ be a separated morphism where $Y$ is affine. Show that if $U, V \subseteq X$ are open affine subschemes, then $U \cap V$ is also affine. Show that this is not true without the separated condition.

*Solution.* □

*Exercise* A.25. A morphism $f\colon X \to Y$ is said to be *finite* if $Y$ can be covered with open affine subschemes $U_i = \operatorname{Spec} A_i$ such that for each $i$ the inverse image $f^{-1}U_i$ is affine, say $\operatorname{Spec} B_i$, and such that $B_i$ is a finitely generated $A_i$-module.

Show that a finite morphism is proper.

*Solution.* Finite morphisms are of finite type, so that, like in exercise A.23, we only need to show properness for an affine cover. Let $U_i$ be a cover as in the definition. Then we need to show that for every valuation ring $R$ with fraction field $K$ and every diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & f^{-1}U_i \arrow[d] \\ \operatorname{Spec} R \arrow[r] & U_i \end{tikzcd}$$

there exists exactly one morphism from $\operatorname{Spec} R$ to $f^{-1}U_i$ fitting into the diagram. Since all schemes are affine, we can go to commutative algebra:

$$\begin{tikzcd} K & B_i \arrow[l, "\beta"'] \arrow[dl, dashed, "g"] \\ R \arrow[u, hook] & A_i \arrow[l, "\alpha"] \arrow[u] \end{tikzcd}$$

Uniqueness of $g$ is clear, so we only need to prove existence. Since $B_i$ is a finitely generated $A_i$-module, $\beta(B_i)$ is a finitely generated $\alpha(A_i)$-module. Hence it is integral over $\alpha(A_i)$ [Eis95, Corollary 4.6]. But $R$ is integrally closed in $K$ [Eis95, Exercise 11.1b] and contains $\alpha(A_i)$. So $\beta(B_i) \subseteq R$ as required. $\square$

*Exercise* A.26. Let $Y = \operatorname{Spec} A$. Show that $\mathbb{P}^n_Y \cong \operatorname{Proj} A[t_0, \ldots, t_n]$.

*Solution.* TBA $\square$

## A.3. Sheet 3

*Exercise* A.27. Show that a schemes $X$ is affine if and only if there are $b_1, \ldots, b_n \in \mathcal{O}_X(X)$ such that $D(b_i)$ is affine and the ideal generated by all the $b_i$ is $\mathcal{O}_X(X)$. (see [Har77, exercise II.2.17])

*Solution.* TBA $\square$

*Exercise* A.28. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Show that $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{G}_x$.

*Solution.* TBA $\square$

*Exercise* A.29. Let $X = \operatorname{Spec} A$ be an affine scheme. Show that $\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ for any $A$-modules $M$, $N$.

*Solution.* TBA $\square$

*Exercise* A.30. Let $X$ be a scheme and let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. Show that of two of the sheaves are quasi-coherent, then the third one is also quasi-coherent.

*Solution.* If $\mathcal{G}$ and $\mathcal{E}$ are quasi-coherent, we can proceed as in Theorem 2.80. For the other two we need the following observation: If $\mathcal{F}$ is quasi-coherent and $U \subseteq X$ is affine, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{E}(U) \to 0$$

is exact. This can be proven either elementarily as in [Har77, Proposition II.5.6] or from Theorem 3.15.

Suppose $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent. Then the cokernel $\mathcal{E}$ is quasi-coherent by the exact same argument as in Theorem 2.80 (using the above observation).

If $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, we get a commutative diagram (for $U \subseteq X$ affine, using Lemma 2.76)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \widetilde{\mathcal{F}(U)} \arrow[r] \arrow[d, dashed, "\cong"] & \widetilde{\mathcal{G}(U)} \arrow[r] \arrow[d] & \widetilde{\mathcal{E}(U)} \arrow[r] \arrow[d, dashed, "\cong"] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{F}|_U \arrow[r] & \mathcal{G}|_U \arrow[r] & \mathcal{E}|_U \arrow[r] & 0
\end{tikzcd}$$

By the five lemma, $\widetilde{\mathcal{G}(U)} \to \mathcal{G}|_U$ is an isomorphism. Hence $\mathcal{G}$ is quasi-coherent. $\square$

*Exercise* A.31. Let $X = \operatorname{Proj} S$, where $S = \bigoplus_{d \geq 0} S_d$ is a graded ring that is generated by $S_1$ over $S_0$. Prove that for every integer $n$, $\mathcal{O}_X(n)$ is an invertible sheaf.

*Solution.* TBA $\square$

*Exercise* A.32. Let $X = \mathbb{P}^n_k = \operatorname{Proj} k[t_0, \ldots, t_n]$ where $k$ is a field, and let $Y$ be the closed sub-$k$ scheme defined by the ideal generated by a homogeneous polynomial $F$ of degree $d$. Show that the ideal sheaf $\mathcal{I}_Y$ of $Y$ is isomorphic to $\mathcal{O}_X(-d)$.

*Solution.* Let $S = k[t_0, \dots, t_n]$. Then the map $p \to pF$ is an isomorphism of graded $S$-modules $S(-d) \to (F) \subseteq S$. Thus the ideal sheaf $\widetilde{(F)}$ defining $Y$ is isomorphic to $\widetilde{S(-d)} = \mathcal{O}_X(-d)$. $\square$

## A.4. Sheet 4

*Exercise* A.33. Let $f\colon X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a flasque sheaf in $\mathfrak{Sh}(X)$. Show that $f_*\mathcal{F}$ is flasque.

*Proof.* Let $V \subseteq U \subseteq Y$ be open subsets. Then the restriction map $f_*\mathcal{F}(U) \to f_*\mathcal{F}(V)$ is just $\mathcal{F}(f^{-1}(U)) \to \mathcal{F}(f^{-1}(V))$, which is surjective by assumption. $\square$

*Exercise* A.34. Let $X$ be a closed subset of a topological space $Y$ and $f\colon X \to Y$ the inclusion. Prove that for any sheaf $\mathcal{F}$ on $X$ we have $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$ for every $p$.

*Proof.* Let $I^\bullet$ be a flasque resolution of $\mathcal{F}$ on $X$. Then $f_*I^\bullet$ is a flasque resolution of $f_*\mathcal{F}$: for $p \in X$, $(f_*\mathcal{F})_p = \mathcal{F}_p$ and for $p \notin X$, sheaf $(f_*\mathcal{F})_p = 0$, and hence the complex stays exact. Since $\mathcal{F}(X) = f_*\mathcal{F}(Y)$, the cohomology groups are the same. $\square$

*Exercise* A.35. Let $X$ be a Noetherian scheme and let $\mathfrak{Q}(X)$ be the category of quasi-coherent schemes on $X$. Show that $\mathfrak{Q}(X)$ has enough injectives.

*Proof.* Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. We need to embed it into a quasi-coherent injective (in $\mathfrak{Q}(X)$) sheaf $I$. Fix a finite cover of $X$ by open affine subschemes $U_i$ and let $\iota_i$ be the inclusions $U_i \hookrightarrow X$.

There exist $\mathcal{O}_X(U_i)$-modules $M_i$ such that $\mathcal{F}|_{U_i} = \widetilde{M_i}$. Since the category of $\mathcal{O}_X(U_i)$-modules has enough injectives, we can find injective modules $I_i$ such that $M_i \hookrightarrow I_i$. Set $I = \bigoplus_i \iota_{i*}\widetilde{I}_i$. We have injections $\widetilde{M_i} \hookrightarrow \widetilde{I}_i$ and thus morphisms $\mathcal{F} \to \iota_{i*}\widetilde{I}_i$. Taking the direct sum over $i$ gives an injective morphism $\mathcal{F} \to I$.

We need to show that $I$ is injective. Let $0 \to \mathcal{G} \to \mathcal{H}$ be an injection of quasi-coherent sheaves on $X$ and assume that we have a morphism $\mathcal{G} \to I$. Any morphism $\mathcal{G} \to \iota_{i*}\widetilde{I}_i$ corresponds to a morphism $\mathcal{G}|_{U_i} \to \widetilde{I}_i$. As $I_i$ is injective, we can extend this morphism to a morphism $\mathcal{H}|_{U_i} \to \widetilde{I}_i$. Taking the direct sum over $i$ of these morphisms, we obtain a morphism $\mathcal{H} \to I$ which extends $\mathcal{G} \to I$. Thus $I$ is indeed injective. $\square$

*Exercise* A.36. A morphism $f\colon X \to Y$ is said to be affine if for any open affine subscheme $U \subseteq Y$, $f^{-1}U$ is affine. Now let $f\colon X \to Y$ be an affine morphism of separated Noetherian schemes. Prove that for any quasi-coherent sheaf $\mathcal{F}$ on $X$, we have $H^p(Y, f_*\mathcal{F}) \cong H^p(X, \mathcal{F})$ for every $p$.

*Solution.* We can compute sheaf cohomology of $\mathcal{F}$ and $f^*\mathcal{F}$ using Čech cohomology. So let $\mathfrak{U}$ be a finite open affine cover of $Y$. The $\mathfrak{U}' = \{f^{-1}U : U \in \mathfrak{U}\}$ is a cover of $X$ by open affine subschemes. Note that $C^p(\mathfrak{U}', \mathcal{F}) = C^p(\mathfrak{U}, f_*\mathcal{F})$. Hence the Čech cohomology groups of $\mathcal{F}$ on $X$ and of $f_*\mathcal{F}$ on $Y$ agree. $\square$

*Exercise* A.37. Let $X$ be a closed subscheme of $\mathbb{P}^2_k$ defined by the ideal of a homogeneous polynomial $F$ of degree $d$ where $k$ is a field. Show that $\dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim_k H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$.

*Solution.* By exercise A.32, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_k}(-d) \to \mathcal{O}_{\mathbb{P}^2_k} \to i_*\mathcal{O}_X \to 0,$$

where $i\colon X \to \mathbb{P}^2_k$ is the closed immersion. The corresponding long exact sequence starts with

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \arrow[r] & H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \arrow[r] & H^0(\mathbb{P}^2_k, i_*\mathcal{O}_X(d)) \arrow[dll, rounded corners] \\
& H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \arrow[r] & H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \arrow[r] & H^1(\mathbb{P}^2_k, i_*\mathcal{O}_X(d)) \arrow[dll, rounded corners] \\
& H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \arrow[r] & H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \arrow[r] & H^2(\mathbb{P}^2_k, i_*\mathcal{O}_X(d))
\end{tikzcd}$$

We have already calculated most of these groups in Theorem 3.28. The only two non-vanishing segments of the sequence are

$$0 \to H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \to H^0(\mathbb{P}^2_k, i_*\mathcal{O}_X) \to 0,$$

$$0 \to H^1(\mathbb{P}^2_k, i_*\mathcal{O}_X) \to H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \to 0.$$

So (using Exercise A.34 and again Theorem 3.28) we have

$$H^0(\mathbb{P}^2_k, \mathcal{O}_X) \cong H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \cong k$$

and

$$H^1(\mathbb{P}^2_k, \mathcal{O}_X) \cong H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \cong H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d-3)).$$

The last cohomology group is isomorphic to the $k$-vector space of monomials of degree $d-3$ in 3 variables, which has dimension $\binom{d-3+2}{2} = \frac{(d-1)(d-2)}{2}$. $\square$

*Exercise* A.38. Let $X$ be a topological space, $\mathfrak{U} = (U_i)_{i \in I}$ a finite open cover, and $\mathcal{F}$ a sheaf on $X$. Assume that for each $p \geq 0$ and each $i_0 < \cdots < i_p$ we have $H^l(U_{i_0,\ldots,i_p}, \mathcal{F}|_{U_{i_0,\ldots,i_p}}) = 0$ whenever $l > 0$. Show that there are isomorphisms $H^l(X, \mathcal{F}) \cong \check{H}^l(\mathfrak{U}, \mathcal{F})$ for any $l \geq 0$.

*Solution.* We will work along the lines of the proof of theorem 3.26.

For each $i \in I$, there exists an injective sheaf $I_i$ on $U_i$ such that $\mathcal{F}|_{U_i}$ injects into $I_i$. Let $\mathcal{G}_i$ be the direct image of $I_i$ under the inclusion map $U_i \hookrightarrow X$. Set $\mathcal{G} = \prod \mathcal{G}_i$. Then $\mathcal{G}$ is a flasque sheaf on $X$ and $\mathcal{F}$ injects into $\mathcal{G}$. Let $\mathcal{H} = \mathcal{G}/\mathcal{F}$. So we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.$$

Consider any sequence $i_0 < i_1 < \cdots < i_p$. The long exact sequence of cohomology gives

$$0 \to \mathcal{F}(U_{i_0,\ldots,i_p}) \to \mathcal{G}(U_{i_0,\ldots,i_p}) \to \mathcal{H}(U_{i_0,\ldots,i_p}) \to 0$$

since $H^1(U_{i_0,\ldots,i_p}, \mathcal{F}|_{U_{i_0,\ldots,i_p}}) = 0$. Since also all higher cohomology groups of the flasque sheaf $\mathcal{G}$ vanish, the same long exact sequence shows that $H^l(U_{i_0,\ldots,i_p}, \mathcal{H}|_{U_{i_0,\ldots,i_p}}) = 0$ for all $l \geq 1$, i.e. that the assumption of the theorem also holds for $\mathcal{H}$. The direct sum of all these sequences gives an exact sequence

$$0 \to C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{U}, \mathcal{G}) \to C^\bullet(\mathfrak{U}, \mathcal{H}) \to 0$$

By theorem 1.37 we get a long exact sequence.

$$\cdots \to \check{H}^l(\mathfrak{U}, \mathcal{F}) \to \check{H}^l(\mathfrak{U}, \mathcal{G}) \to \check{H}^l(\mathfrak{U}, \mathcal{H}) \to \check{H}^{l+1}(\mathfrak{U}, \mathcal{F}) \to \cdots$$

Since $\mathcal{G}$ is flasque, $\check{H}^l(\mathfrak{U}, \mathcal{G}) = 0 = H^l(X, \mathcal{G})$ for all $l \geq 1$ (theorems 3.9 and 3.25). In particular, we have

$$\check{H}^0(\mathfrak{U},\mathcal{F}) \longrightarrow \check{H}^0(\mathfrak{U},\mathcal{G}) \longrightarrow \check{H}^0(\mathfrak{U},\mathcal{H}) \longrightarrow \check{H}^1(\mathfrak{U},\mathcal{F}) \longrightarrow 0$$

$$H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{G}) \longrightarrow H^0(X,\mathcal{H}) \longrightarrow H^1(X,\mathcal{F}) \longrightarrow 0$$

We already know that $\check{H}^0(\mathfrak{U},\mathcal{F}) = H^0(X,\mathcal{F}) = \mathcal{F}(X)$ (Theorem 3.17; the same is true for $\mathcal{G}$ and $\mathcal{H}$). The maps on the 0-th cohomology groups are just the maps on global sections. So the cokernels are the same, i.e. $\check{H}^1(\mathfrak{U},\mathcal{F}) \cong H^1(X,\mathcal{F})$ and the analogous statement holds for $\mathcal{H}$ (since it also fulfills the assumptions). Parts of the long exact sequence look like

$$0 \longrightarrow \check{H}^l(\mathfrak{U},\mathcal{H}) \longrightarrow \check{H}^{l+1}(\mathfrak{U},\mathcal{F}) \longrightarrow 0$$

$$0 \longrightarrow H^l(X,\mathcal{H}) \longrightarrow H^{l+1}(X,\mathcal{F}) \longrightarrow 0$$

This allows us to do induction to show the result for all $l$. $\square$

# B. Some Mathematicians

The information here is mainly taken from Wikipedia. I tried to add German pronunciations when they where not available in Wikipedia. However, my knowledge of the IPA is very limited, so there might be errors.

- Pierre Cartier [French pronunciation] (1932 (Sedan, France) –)
- Eduard Čech [ˈɛduart ˈtʃɛx] (1893 (Stračov, Bohemia; now in Czech Republic) – 1960 (Prague))
- Alexander Grothendieck [ˌalɛkˈsandɐ ˌɡʀotɛnˈdiːk] (1928 (Berlin, Germany) – )
- Amalie Emmy Noether [ˈnøːtɐ] (1882 (Erlangen, Germany) – 1935 (Bryn Mawr, Pennsylvania, USA))
- Charles Émile Picard [French pronunciation] (1856 (Paris, France) – 1941 (Paris))
- Georg Friedrich Bernhard Riemann [ˈʀiːman] (1826 (Breselenz, Germany) – 1866 (Selasca, Italy))
- Gustav Roch [ʀox] (1839 (Leipzig, Germany) – 1866 (Venice, Italy))
- Jean-Pierre Serre [French pronunciation] (1926 (Bages, France) – )
- André Weil [ãdʀe vɛj] (1906 (Paris, France) – 1998 (Princeton, USA))
- Oscar Zariski (born Oscher Zaritsky) [?] (1899 (Kobrin, Russia; today Belarus) – 1986 (Brookline, Massachusetts, USA))

# C. Changes

1.0.1 – 2010-08-10

- Small error in the proof of Theorem 2.23.
- Typos.

1.0 – 2010-06-11

- Added lemma 2.5 (quasi-compactness off principal open subsets), therefore changing the numbering in chapter 2 by one.
- Corrected many errors. In particular the statement of Theorem 3.15 was not entirely as in the lectures.

0.3.2 – 2010-06-04

- Some solutions for the example sheets.
- Some typographical improvements.

0.3.1 – 2010-05-08

- Some solutions for the example sheets.
- Some small corrections an improvements (removed an incorrect remark).

0.3 – 2010-04-24

- Added the missing assumption "Noetherian" to Theorem 3.13.
- Split out two lemmas from Theorem 3.9. Note that this changed all theorem numbers in Chapter 3.
- Added a missing sheafification in the proof of Theorem 3.6.
- Minor corrections and improvements.

0.2 – 2010-02-26

- Changed the license to CC BY-SA (somehow forgot to add the Share Alike part).
- Construction of the fibred product for affine schemes (theorem 2.33). Examples for fibres.
- One example of a separated morphism from lectures.
- Added information about some of the mathematicians involved (if you can provide better pronunciation guides or want to write biographies, please write me an email).
- Added the remaining part of the proof of theorem 2.91.

# Bibliography


[Con78] John B Conway. *Functions of One Complex Variable*, volume 11 of *Graduate Texts in Mathematics*. Springer, New York, 1978.

[EH00] David Eisenbud and Joe Harris. *The Geometry of Schemes*. Springer, New York, 2000.

[Eis95] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer, 1995.

[GD60] Alexander Grothendieck and Jean Dieudonné. *Éléments de géométrie algébrique: I. Le langage des schémas*. Publications mathématiques de l'I.H.É.S., 1960. Available from: `http://www.numdam.org/item?id=PMIHES_1960__4__5_0`.

[GD61a] Alexander Grothendieck and Jean Dieudonné. *Éléments de géométrie algébrique: II. Étude globale élémentaire de quelques classes de morphismes*. Publications mathématiques de l'I.H.É.S., 1961. Available from: `http://www.numdam.org/item?id=PMIHES_1961__8__5_0`.

[GD61b] Alexander Grothendieck and Jean Dieudonné. *Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, Première partie*. Publications mathématiques de l'I.H.É.S., 1961. Available from: `http://www.numdam.org/item?id=PMIHES_1961__11__5_0`.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, New York, 1977.

[Liu02] Qing Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford University Press, New York, 2002.

# List of Notation

# Index