Differential Geometry

Lecture Notes[1]

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[1]Following the Part III lecture of the same name by Dr. J. Ross. Michaelmas term 2009, Cambridge.

These notes are not endorsed by Julius Ross. They have not been carefully proofread and might therefore contain lots of errors. Please notify me of any you find at me@caramdir.at.

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Preliminaries

Notation The symbol $A \subseteq_o B$ means A is an open subset of B. This notation is not used in the lecture, but is useful to avoid the constant reoccurring "where U is an open subset of $\mathbb{R}^n$".

1 Manifolds

Definition 1.1. A *chart* on a set M is a bijection $x\colon U \to \mathbb{R}^n$ from $U \subseteq M$ to some open set $x(U) \subseteq \mathbb{R}^n$.

An *atlas* on M is a collection of charts (x_α, U_α) such that M is covered by the U_α and for all α, β: $x(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$ and the *transition map*

$$x_\beta \circ x_\alpha^{-1} \colon x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$$

is smooth (i.e. C^∞).

Two atlases are *compatible*, if their union is an atlas. This gives an equivalence relation on the set of atlases. An equivalence class of this relation is called a *differentiable structure*. A *manifold* is a set with a differentiable structure.

Remarks 1.2.

- Given a chart (x, U) and a function $f \colon M \to \mathbb{R}$. We abuse notation and confuse $f|_U$ with $f \circ x^{-1}$. E.g. if $x(p) = (x^1, \ldots, x^n)$ we write $f(x^1, \ldots, x^n)$ for $f(p)$.
- If $x(p) = 0$, we say the chart (x, U) is *centred* at p.
- A differentiable structure defines a topology on M by $V \subseteq M$ is open if and only if $x_\alpha(V \cap U_\alpha)$ is open for all charts (x_α, U_α). (In particular, all U_α are open.) We always assume that this topology is Hausdorff and second countable (i.e. has a countable basis).
- Given charts $x_\alpha \colon U_\alpha \to \mathbb{R}^n$ and $x_\beta \colon U_\beta \to \mathbb{R}^n$ with $U_\alpha \cap U_\beta \neq \emptyset$, the transition map gives a homeomorphism from some nonempty subset of $\mathbb{R}^n$ to the same in $\mathbb{R}^m$. By a theorem in topology this implies $m = n$. Hence there is a well-defined notion of *dimension* $\dim_p M$. If M is connected, then $\dim M$ is well-defined.

Examples 1.3.

- $M = \mathbb{R}^n$ with the single chart $x \colon \mathbb{R}^n \to \mathbb{R}^n$, $x(t) = t$.
- $M = \mathbb{R}$. Let $\mathcal{A}$ be the atlas given by the usual differential structure and let the atlas $\mathcal{A}'$ be given by the single chart $y \colon \mathbb{R} \to \mathbb{R}$, $y(t) = t^3$. Since $t \to t^{\frac{1}{3}}$ is not smooth, the atlases $\mathcal{A}$ and $\mathcal{A}'$ are not compatible.
- $M = S^n$ with stereographic projection.
- If U is an open subset of any manifold, then U is a manifold with charts $U_\alpha \cap U$ for charts U_α of M.
- $M_n = \{n \times n\text{-matrices with real values}\} \cong \mathbb{R}^{n^2}$;
  $\mathrm{GL}_n = \{A \in M_n : \det A \neq 0\}$ is open in M_n and so is a manifold.
- *Real projective space* $\mathbb{RP}^n$ with the usual charts.
- $\mathbb{C} \cong \mathbb{R}^2$ is a manifold as is $M_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{C})$ and $\mathbb{PC}^n$.

Definition 1.4. Let M and N be manifolds. A function $F \colon M \to N$ is *smooth* at $p \in M$, if there exists a chart (x, U) of M with $p \in U$ and a chart (y, V) of N with $F(p) \in V$ such that $y \circ F \circ x^{-1}$ is smooth at $x(p)$. We set

$$C^\infty(M) = \{f \colon M \to \mathbb{R} \text{ smooth}\}.$$

A function $F \colon M \to N$ is a *diffeomorphism* if it is smooth, bijective and has a smooth inverse. We set

$$\mathrm{Diff}(M) = \{F \colon M \to M \text{ diffeomorphism}\}.$$

Note that if a function is smooth with respect to some charts, it is smooth with respect to every chart (by the chain rule).

Example 1.5. Let $M = \mathbb{R}$ with the usual differential structure and $N = \mathbb{R}$ with the atlas $\mathcal{A}'$ from above (i.e. given by $y(t) = t^3$). Then $F \colon M \to N$, $F(t) = t^{1/3}$ is a diffeomorphism.

2 Tangent space

Notation. For $U \subseteq_o \mathbb{R}^n$, $f\colon U \to \mathbb{R}$, $p \in U$ and $v \in \mathbb{R}^n$ the directional derivative is

$$D_v|_p(f) = \lim_{h\to 0} \frac{f(p+vh)-f(p)}{h}.$$

Definition 2.1. Let M be a manifold and $p \in M$. A *tangent vector* to M at p is a derivation on the algebra of germs of M at p.

Example 2.2. Suppose $\gamma\colon (-\varepsilon,\varepsilon) \to M$ is smooth with $\gamma(0) = p$. Define $\dot\gamma(0)(f) = \frac{d}{dt}\big|_{t=0} f \circ \gamma$. Then $\dot\gamma(0)$ is a tangent vector at p.

Notation. Let $x^1, \dots, x^n$ be coordinates around p (i.e. $x = (x^1, \dots, x^n)$, (x, U) is a chart). Set

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \left.\frac{\partial f}{\partial x^i}\right|_p := D_{e_i}|_{x(p)}(f \circ x^{-1}).$$

(This is the above for $\gamma(t) = (0, \dots, 0, t, 0, \dots, 0)$.)

Definition 2.3. The set of tangent vectors at $p \in M$ forms a vector space, called the *tangent space to M at p* and denoted T_pM.

Definition 2.4. Let $F\colon M \to N$ be a smooth map and $p \in M$. The *derivative* of F at p is the linear map

$$DF|_p : T_pM \to T_{F(p)}M \text{ given by } DF|_p(v)(f) = v(f \circ F).$$

It is also denoted by $(F_*)|_p$.

The derivative indeed is well-defined (exercise).

Theorem 2.5 (Chain Rule)**.** *If $f\colon M \to M'$ and $G\colon M' \to N$ are smooth functions of manifolds, then*

$$D(G \circ F)|_p = DG|_{F(p)} \circ DF|_p.$$

Proof. Let $v \in T_pM$

$$D(G \circ F)|_p(v)(f) = v(f \circ G \circ F) = DF|_p(v)(f \circ G) = DG|_{F(p)}(DF|_p(v))(f).$$

□

Corollary 2.6. *If F is a diffeomorphism, then $DF|_p$ is an isomorphism with $(DF|_p)^{-1} = DF^{-1}|_{F(p)}$.*

Theorem 2.7. *Given coordinates $x^1, \dots, x^n$ around p, $\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p$ are a basis of T_pM.*

Proof. First, we consider the case that $p = 0 \in U \subseteq_o \mathbb{R}^n$.

Claim: Suppose that U is convex. Given $f\colon U \to \mathbb{R}$ smooth, there exist $h_i\colon U \to \mathbb{R}$ such that $f(r_1, \dots, r_n) = f(0, \dots, 0) + \sum r_i h_i(r_1, \dots, r_n)$ and $h_i(0) = D_{e_i} f|_0$.

Indeed, fix $\mathsf{r} \in U$ and set $G(t) = f(t\mathsf{r})$, $t \in [0,1]$. Then, by the fundamental theorem of calculus and the chain rule,

$$f(\mathsf{r}) - f(0) = G(1) - G(0) = \int_0^1 G'(t)dt = \int_0^1 \sum_{i=1}^n D_{e_i} f(t\mathsf{r}) r_i dt.$$

Set $h_i(\mathsf{r}) = \int_0^1 \sum_{i=1}^n D_{e_i} f(t\mathsf{r}) dt$.

Next we will show that $D_{e_1}|_0, \ldots, D_{e_n}|_0$ is a basis for $T_0\mathbb{R}^n$. Let $v \in T_0\mathbb{R}^n$. Given f, write $f(\mathsf{r}) = f(0) + \sum_i r_i h_i(\mathsf{r})$ as above. Then

$$v(f) = \underbrace{v(f(0))}_{=0,\text{ since } f(0) \text{ is constant}} + v\left(\sum r_i h_i(\mathsf{r})\right) = \sum v(r_i)|_0 \, h_i(0) + r_i|_0 \, v(h_i) = \sum_i v(r_i)|_0 \, D_{e_i} \, f|_0 \, .$$

Set $a_i = v(r_i)|_0$. Then $v = \sum a_i \, D_{e_i}|_0$ and hence the $D_{e_i}|_0$ span the tangent space. Because of $D_{e_i}|_0 \,(r_j) = \delta_{ij}$, they are also linearly independent.

For the general case we have to show that $\frac{\partial}{\partial x^i}\big|_p$ are a basis of T_pM. Consider the diffeomorphism $x\colon U \to x(U)$. Without loss of generality $x(p) = 0$. Then $Dx|_p : T_pM \to T_0\mathbb{R}$ is an isomorphism and one checks that $Dx|_p \,(\frac{\partial}{\partial x^i}\big|_p) = D_{e_i}|_p$. □

Concretely this means that for $v \in T_pM$, $v = \sum_i v(x^i) \, \frac{\partial}{\partial x^i}\big|_p$. [If $v = \sum_i a^i \, \frac{\partial}{\partial x^i}\big|_p$, then $v(x^j) = \sum_i a^i \delta_{ij} = a^j$.]

Corollary 2.8 (Transformation Law)**.** *Suppose $y^1, \ldots, y^n$ are another set of coordinates at p. Then*

$$\left.\frac{\partial}{\partial y^j}\right|_p = \sum_i \left.\frac{\partial x^i}{\partial y^j}\right|_p \left.\frac{\partial}{\partial x^i}\right|_p .$$

In particular, for $v \in T_pM$ with $v = \sum_i a^i \, \frac{\partial}{\partial x^i}\big|_p = \sum_j b^j \, \frac{\partial}{\partial y^j}\Big|_p$, we have $a^i = \sum_j b^j \, \frac{\partial x^i}{\partial y^j}\Big|_p$.

Proof. $\sum_i a^i \, \frac{\partial}{\partial x^i}\big|_p = \sum_j b^j \sum_i \frac{\partial x^i}{\partial y^j}\Big|_p \, \frac{\partial}{\partial x^i}\big|_p \, .$ □

From any linear function $f\colon U \to \mathbb{R}$ ($p \in U \subseteq_o M$) we get a linear map

$$Df|_p \colon T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$$

and hence an element the dual space $T_p^*M = (T_pM)^*$. It is denoted $df|_p$.

Lemma 2.9.

1. $dx^1\big|_p, \ldots, dx^n\big|_p$ *is the dual basis to* $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p$.
2. $df|_p = \sum_i \frac{\partial f}{\partial x^i}\Big|_p \, dx^i\big|_p$.

Hence $df|_p$ carries the same information as the gradient.

Proof.

1. Consider $g\colon \mathbb{R} \to \mathbb{R}, t \mapsto t$:

$$dx^i\big|_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right)(g) = \left.\frac{\partial}{\partial x^j}\right|_p (g \circ x^i) = \left.\frac{\partial}{\partial x^j}\right|_p (x^i) = \delta_{ij}.$$

2. $df|_p \left(\frac{\partial}{\partial x^j}\big|_p \right) = \frac{\partial}{\partial x^j}\big|_p (f) = \frac{\partial f}{\partial x^i}\Big|_p$. □

3 The tangent bundle

Definition 3.1. Let M be a manifold. Define the *tangent bundle* TM of M as follows: As a set $TM = \coprod_{p \in M} T_pM$. For every chart (x, U) of M define a chart (t_U, TU) on TM by $TU = \coprod_{p \in U} T_pM$ and

$$t_U \colon TU \to x(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}, \qquad v_p \in T_pM \mapsto (x(p), (a^1(p), \ldots, a^n(p))^t),$$

where $v_p = \sum_i a^i(p) \, \frac{\partial}{\partial x^i}\big|_p$.

This gives a differential structure on TM, which makes TM into a manifold of dimension $2\dim M$: Given another chart (y, V) the transition map is given by

$$t_U t_V^{-1} \colon y(U\cap V)\times \mathbb{R}^n \to x(U\cap V)\times\mathbb{R}^n, \qquad (\alpha, (b^1,\dots,b^n)^t) \mapsto (xy^{-1}(\alpha), (a^1,\dots,a^n)^t),$$

where $a^i = \sum_j b^j \frac{\partial}{\partial x^i} y^j$. So they are smooth. Also this gives a topology on TM that is Hausdorff and second countable (exercise).

Lemma 3.2. *The map $\pi\colon TM \to M$ mapping $v_p \in T_pM$ to p is smooth.*

Proof. Given $p \in M$ and a chart (x, U) around p we have

$$x \circ \pi \circ t_U^{-1}(\alpha, (a^1,\dots,a^n)^t) = \alpha.$$ □

Consider a function $X\colon U \to TU$ with $\pi \circ X = \mathrm{id}_U$. Then $X = \sum_i a^i \frac{\partial}{\partial x^i}$ for some $a^i\colon U \to \mathbb{R}$. We see that X is smooth if and only if all a^i are smooth.

Definition 3.3. A smooth map $X\colon U \to TM$ with $\pi\circ X = \mathrm{id}_U$ is called a *vector field* on U. We will often write X_p for $X(p)$. The $C^\infty(U)$-module of vector fields on U is denoted $\mathrm{Vect}(U)$. Here multiplication is pointwise, i.e. $(fX)(p) = f(p)X_p$. Further, $\mathrm{Vect}(U)$ acts on $C^\infty(U)$ by $(Xg)(p) = X_p(g)(p) =: X_p(g)$. (Locally $Xg = \sum a^i \frac{\partial g}{\partial x^i}$, so it is smooth.)

Definition 3.4. Let $F\colon M \to N$ be smooth. Then define

$$DF\colon TM \to TN, \qquad v_p \in T_pM \mapsto DF|_p(v_p) \in T_{F(p)}N.$$

Proposition 3.5. *DF is smooth and*

$$\begin{array}{ccc} TM & \xrightarrow{DF} & TN \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{F} & N \end{array}$$

commutes.

Proof. Let $\dim M = m$ and $\dim N = n$. Pick charts (x, U) around $p \in M$ and (y, V) around $F(p) \in N$. Say $F = (F^1,\dots,F^n)$ where $F^i = F^i(x^1,\dots,x^m)$ are smooth. Then $DF|_p\left(\frac{\partial}{\partial x^i}\big|_p\right) = \sum_j a^j \frac{\partial}{\partial y^j}\Big|_{F(p)}$ with

$$a^j = DF|_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right)(y^j) = \frac{\partial}{\partial x^i}\bigg|_p (y^j\circ F) = \frac{\partial F^j}{\partial x^i}\bigg|_p.$$

In other words, with respect to the basis $\frac{\partial}{\partial x^i}\big|_p$ of T_pM and $\frac{\partial}{\partial x^j}\big|_{F(p)}$ of $T_{F(p)}N$, $DF|_p$ is given by the matrix $\left(\frac{\partial F^j}{\partial x^i}\big|_p\right)_{j,i}$. [By abuse of notation this matrix is also denoted $DF|_p$.] Thus

$$t_V \circ DF \circ t_U^{-1}(\alpha, (b^1,\dots,b^n)^t) = \big(yFx^{-1}(\alpha), \underbrace{DF|_p}_{\text{the matrix}}(b^1,\dots,b^n)^t\big),$$

which is smooth. □

Warning. While it is possible to "push forward" individual tangent vectors via DF, for a vector field $X \in \mathrm{Vect}(M)$, $DF(X)$ does not make sense in general, as for $q \in N$ there might be more than one point in $F^{-1}(q)$.

On the other hand, if F is a diffeomorphism, then $DF(X)$ makes sense:

$$(F_*X)\big|_q = DF(X)\big|_q = DF|_{F^{-1}(q)}\,(X_{F^{-1}(q)}).$$

Remark 3.6. We can think of vector fields a derivations on the space of smooth functions:

$$\big(X(gh)\big)(p) = X_p(gh) = g(p)X_p(h) + h(p)X_p(g),$$

i.e. $X(gh) = gX(h) + hX(g)$.

Proposition 3.7. *Given any derivation $\mathcal{X}\colon C^\infty(M) \to C^\infty(M)$, there exists a unique vector field X on M such that $\mathcal{X}(g)(p) = X_p(g)$ for all $g \in C^\infty(M)$. This defines an isomorphism between derivations on $C^\infty(M)$ and* $\mathrm{Vect}(M)$.

For the proof the following construction is needed, that will also come in handy later on.

Lemma 3.8 (Bump Functions)**.** *Let $p \in U \subseteq_o M$. Then there exists an open subset $p \in U' \subseteq U$ and a function $\varphi \in C^\infty(M)$ such that $\varphi \equiv 1$ on U' and* $\mathrm{supp}(\varphi) \subseteq U$.

Proof. Define:

$$\alpha(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \le 0 \end{cases}$$
$$\beta(t) = \frac{\alpha(t)}{\alpha(t) + \alpha(1-t)}$$
$$\gamma(t) = \beta(2-t)\beta(2+t).$$

Given $\varepsilon > 0$, set $h_\varepsilon(x) = \gamma\left(\frac{\|x\|}{\varepsilon}\right)$, $x \in \mathbb{R}^n$. If (x, U) is a chart, set

$$\varphi(q) = \begin{cases} h_\varepsilon(x(q)), & q \in U \\ 0, & q \notin U \end{cases}$$

for some ε small enough. □

Proof of proposition. Suppose $f|_W = g|_W$ for some $p \in W \subseteq_o M$. We need to show that $\mathcal{X}(p)(f) = \mathcal{X}(p)(g)$. Pick a bump function φ such that $\varphi \equiv 1$ on some $p \in W'$ and $\mathrm{supp}\,\varphi \subseteq W$. Then $\varphi f = \varphi g$ on M. Hence

$$\mathcal{X}(\varphi f) = \mathcal{X}(\varphi g)$$
$$\varphi\mathcal{X}(f) + f\mathcal{X}(\varphi) = \varphi\mathcal{X}(g) + g\mathcal{X}(\varphi).$$

Thus $\mathcal{X}(f)(p) = \mathcal{X}(g)(p)$ and so X_p is well-defined on germs. As $\mathcal{X}$ is a derivation, X_p is one. So $X_p \in T_pM$.

To check that $p \mapsto X_p$ is smooth, introduce local coordinates $X_p = \sum \alpha^i(p)\, \frac{\partial}{\partial x^i}\big|_p$. Then

$$\alpha^i(p) = X_p(x^i) = \mathcal{X}(\varphi x^i)\big|_p$$

for a suitable bump function φ. So α^i and hence X is smooth. □

A section $X\colon M \to TM$ is smooth if and only if $X(f)$ is smooth for all $f \in C^\infty(M)$.

If $X, Y \in \mathrm{Vect}(M)$, then XY does not need to be a vector field as

$$X(Y(gf)) = X(gY(f) + fY(g)) = X(g)Y(f) + gX(Y(f)) + X(f)Y(g) + fX(Y(g))$$

does not give a derivation. However, $XY - YX$ does give a derivation.

Definition 3.9. Let X and Y be vector fields on M. Then the *Lie bracket* of X and Y is the vector field $[X, Y] = XY - YX$.

Proposition 3.10.

1. $[-,-]$ *makes* $\mathrm{Vect}(M)$ *into a (possibly infinite dimensional) Lie algebra, i.e.*
   a) $[-,-]$ *is bilinear and anticommutative* $([X, Y] = -[Y, X])$*;*
   b) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(*Jacobi identity*)*
2. *If* $f, g \in C^\infty(M)$*, then* $[fX, fY] = fg[X, Y] + f(Xg)Y - g(Yf)X$*.*
3. *If* $x^1, \ldots, x^n$ *are coordinates, then* $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ *for all* i, j*.*

4 Lie groups

Definition 4.1. A *Lie group* G is a manifold G that is also a group such that the maps

$$m\colon G \times G \to G,\ (g, h) \mapsto gh \quad \text{and} \quad i\colon G \to G,\ g \mapsto g^{-1}$$

are smooth.

Example 4.2. $GL(n, \mathbb{R})$, $S^1 \subseteq \mathbb{C}$ and $S^3 \subset \mathbb{H}$ (where $\mathbb{H}$ are the quaternions and S^3 is thought of as the unit quaternions) are Lie groups.

Given $g \in G$, consider $L_g\colon G \to G$, $h \mapsto gh$, which is smooth with smooth inverse $L_{g^{-1}}$. Then $DL_g|_e : T_eG \to T_gG$ is an isomorphism.

Definition 4.3. The *Lie algebra* of the Lie group G is $\mathfrak{g} = T_eG$.

To justify this definition, we need to define a Lie algebra structure on $\mathfrak{g}$. Recall that if $F\colon M \to M$ is a diffeomorphism and X a vector field on M, then we can define another vector field on M by $(F_*X)|_p = DF|_{F^{-1}(p)}\,(X|_{F^{-1}(p)}) \in T_pM$.

Lemma 4.4. $[F_*X, F_*Y] = F_*[X, Y]$.

Proof. exercise ☐

Definition 4.5. A vector field X on a Lie group G is called *left-invariant* if $(L_g)_*\, X = X$ for all $g \in G$. The space of all left-invariant vector fields on G is denoted $\mathrm{Vect}_L(G)$.

By the lemma, the Lie bracket of two left-invariant vector fields is again left-invariant.

Proposition 4.6. *For* $\xi \in \mathfrak{g}$ *set define* X_ξ *by* $(X_\xi)_g = DL_g(\xi) \in T_gG$*. The map* $\xi \mapsto X_\xi$ *gives a isomorphism* $\mathfrak{g} \to \mathrm{Vect}_L(G)$*.*

Proof. X_ξ is a vector field: Given $f \in C^\infty(X)$:

$$(X_\xi)_g\,(f) = DL_g(\xi)(f) = \xi(f \circ L_g).$$

Since m is smooth, this is smooth with respect to g (exercise).

Because of

$$(L_g)_*\, X_\xi\big|_h = DL_g|_{g^{-1}h}\,(X_\xi|_{g^{-1}h}) = DL_g\big|_{g^{-1}h}\,(DL_{g^{-1}h}\big|_e\,(\xi)) = DL_h\big|_e\,(\xi) = X_\xi|_h$$

(chain rule), X_ξ is left-invariant.

Conversely, if $X \in \mathrm{Vect}_L(G)$, let $\xi = X_e$. Then $X = X_\xi$. ☐

This gives a Lie algebra structure on $\mathfrak{g}$ by $X_{[\xi,\eta]} = [X_\xi, X_\eta]$.

5 Vector fields and flows

Recall: Given a smooth curve $\gamma\colon (-\varepsilon,\varepsilon) \to M$, we set $\dot\gamma(t_0) = D\gamma|_{t_0}\left(\frac{d}{dt}\big|_{t_0}\right)$ where $\frac{d}{dt}\big|_{t_0} \in T_{t_0}\mathbb{R}$.

Definition 5.1. Let X be a vector field on M.

1. An *integral curve* of X through $p \in M$ is a smooth curve $\gamma\colon T \to M$ (with $I \subseteq_o \mathbb{R}$ an interval with $0 \in I$) such that $\gamma(0) = p$ and $\dot\gamma(t) = X_{\gamma(t)}$ for all $t \in I$.
2. A *flow* of X is a smooth $\alpha\colon I \times U \to M$ (with $0 \in I \subseteq_o \mathbb{R}$ and $U \subseteq_o M$) such that for all $p \in U$, $t \mapsto \alpha(t,p)$ is an integral curve of X through p.

Remark 5.2. Suppose that in local coordinates $x^1,\dots,x^n$, $X = \sum a^i \frac{\partial}{\partial x^i}$ and $p = (p_1,\dots,p_n)$. Then $\gamma = (\gamma_1,\dots,\gamma_n)$ is an integral curve if and only if $\gamma_i(0) = p_i$ and $\dot\gamma_i(t) = a^i(\gamma(t))$ for all i.

Theorem 5.3 (Existence and Uniqueness of Solutions of ODEs)**.** *Let $0 \in U \subseteq_o \mathbb{R}^m$ and $X\colon U \to \mathbb{R}^m$ Lipschitz. Then there exist $\varepsilon > 0$ and $a > 0$ such that there exists a unique flow $\alpha : (-\varepsilon,\varepsilon) \times B_a(0) \to \mathbb{R}^m$ of X (i.e. $\frac{d}{dt}\alpha(t,p) = X(\alpha(t,p))$, $\alpha(0,p) = p$ for all $p \in B_a(0)$.) Furthermore, if X is smooth, then so is α (i.e. the solutions of an ODE depend smoothly on the initial conditions).*

Proof. See Lang, Analysis II, chapter VI. □

Theorem 5.4. *Let X be a vector field on M and $p \in M$. Then there exists a unique maximal integral curve $\gamma\colon I \to M$ of X through p (i.e. if $\widetilde\gamma\colon \widetilde I \to M$ is a another integral curve through p, then $\widetilde I \subseteq I$ and $\gamma|_{\widetilde I} = \widetilde\gamma$).*

Proof. Let $I = \bigcup I_c$ be the union over all integral curves $c\colon I_c \to M$ of X through p. By the existence theorem $I \neq \emptyset$. We want to show uniqueness (even outside of a chart around p). Let $S = \left\{t > 0\colon\ \gamma_1|_{(0,t)} = \gamma_2|_{(0,t)}\right\} \neq \emptyset$, $S \subseteq I_{\gamma_1} \cap I_{\gamma_2}$. Set $t_0 = \sup S$ and assume $t_0 < \min\{\sup I_{\gamma_1}, \sup I_{\gamma_2}\}$.

By continuity, $\gamma_1(t_0) = \gamma_2(t_0)$. The curves $t \mapsto \gamma_1(t+t_0)$, $t \mapsto \gamma_2(t+t_0)$ are integral curves through $\gamma_1(t_0) = \gamma_2(t_0)$. By uniqueness (in the ODE theorem), there is a neighborhood of 0 such that $\gamma_1(t+t_0)$ and $\gamma_2(t+t_0)$ agree. Hence there is a neighborhood of t_0 with $\gamma_1 = \gamma_2$, contradicting the choice of t_0.

Hence $t_0 = \min\{\sup I_{\gamma_1}, \sup I_{\gamma_2}\}$. An analogous statement holds for negative time. So we can patch the curves together to give the maximal curve γ. □

Example 5.5. Let $M = \mathbb{R}^2$ and consider $X = \frac{\partial}{\partial x}$. Let $p = (a,b) \in M$. For a curve $\gamma = (\gamma_1,\gamma_2)$ with $\gamma(0) = (a,b)$ to be an integral curve it must fulfill

$$\frac{\partial}{\partial x}\bigg|_{\gamma(t)} = D\gamma\left(\frac{d}{dt}\bigg|_t\right) = \frac{d\gamma_1}{dt}\,\frac{\partial}{\partial x}\bigg|_{\gamma(t)} + \frac{d\gamma_2}{dt}\,\frac{\partial}{\partial y}\bigg|_{\gamma(t)},$$

and hence

$$\frac{d\gamma_1}{dt} = 1, \qquad \frac{d\gamma_2}{dt} = 0.$$

So $\gamma(t) = (a+t,b)$.

For $M = \mathbb{R}^2 \setminus \{0\}$ and $X = \frac{\partial}{\partial x}$, γ is not defined on all of $\mathbb{R}$. In fact at $p = (-\varepsilon,0)$, it is only defined for $t < \varepsilon$.

Example 5.6. Let $M = \mathbb{R}$, $X = x\frac{\partial}{\partial x}$, $\gamma(0) = p$ and γ an integral curve. Then $\dot\gamma(t) = \gamma(t)$ and so $\gamma(t) = pe^t$. Notice that if $p = 0$, then $\gamma(t) \equiv 0$ is constant.

Theorem 5.7 (Existence of Flows)**.** *Let X be a vector field on M and for $p \in M$ let $\gamma_p\colon I_p \to M$ be the maximal integral curve of X through p. Set $I = \bigcap_{p\in M} I_p$ (possibly $I = \{0\}$) and $\varphi\colon T \times M \to M$, $(t,p) \mapsto \gamma_p(t)$. Write $\varphi_t(p) = \varphi(t,p)$. Then:*

1. $\varphi_0 = \mathrm{id}_M$;

2. φ *is smooth;*

3. $\varphi_{t+s} = \varphi_t \circ \varphi_s$ *wherever this is defined.*

φ *is called the* flow of X*.*

Proof.

1. Obvious as $\gamma_p(0) = p$.

2. Comes from the last part of the ODE theorem.

3. Fix s. Then both $t \mapsto \varphi_t \circ \varphi_s(p)$ and $t \mapsto \varphi_{t+s}(p)$ are the unique integral curve through $\varphi_s(p)$. Hence they are equal. $\square$

Definition 5.8. We say that a vector field X is *complete* if for every point $p \in M$ the maximal integral curve through p has domain $\mathbb{R}$.

Theorem 5.9. *If* $\operatorname{supp}(X)$ *is compact (e.g. if* M *is compact), then* X *is complete.*

Proof. For each $p \in M$ there is an open interval I_p around 0 and an open neighborhood $U_p \subseteq M$ of p such that there is a flow $c_q(t)$ on $I_p \times U_p$.

Say $\operatorname{supp}(X) = \overline{\{p \in M : X_p \neq 0\}} \subseteq K$ compact. Then there exist $p_1, \ldots, p_m \in M$ such that U_{p_i} cover K. Set $I = \bigcap_{i=0}^m I_{p_i}$ (is an open interval). For $t \in I$ define

$$\varphi_t(q) = \begin{cases} c_{p_i}(t) & \text{if } q \in \operatorname{supp}(X),\, q \in U_{p_i} \\ 0 & \text{if } q \notin \operatorname{supp}(X) \end{cases}$$

So φ_t is a flow defined on all of M, but only for $t \in I$.

For the extension to all $t \in \mathbb{R}$ fix t and pick $N \in \mathbb{N}$ such that $\frac{t}{N} \in I$. Set

$$\varphi_t = \underbrace{\varphi_{\frac{t}{N}} \circ \cdots \circ \varphi_{\frac{t}{N}}}_{N \text{ times}} = \left(\varphi_{\frac{t}{N}}\right)^N.$$

This does not depend on the choice of N and gives a flow, i.e. $\varphi_{s+t} = \varphi_s \circ \varphi_t$ (exercise). $\square$

5.1 The Lie derivative

Consider $C^\infty(M)$ as a vector space. Then $\operatorname{Diff}(M)$ acts on $C^\infty(M)$ by $(\varphi, f) \mapsto \varphi^* f := f \circ \varphi$. Let $X \in \operatorname{Vect}(M)$ be a vector field with flow φ_t (t defined in some interval around 0). Then $t \mapsto \varphi_t^* f$ is a path in $C^\infty(M)$ through the point $f \in C^\infty(M)$ at $t = 0$.

Definition 5.10. The *Lie derivative* of f along X is

$$L_X f := \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* f = \lim_{t \to 0} \frac{\varphi_t^* f - f}{t}.$$

Proposition 5.11. $L_X f = X f$*.*

Proof. At $p \in M$ consider the integral curve $c_p(t) = \varphi_t(p)$ through p. By definition $\left.\frac{d}{dt}\right|_{t=0} c_p(t) = X_{c_p(0)} = X_p$. Thus

$$L_X f(p) = \left.\frac{d}{dt}\right|_{t=0} f \circ \varphi_t(p) = \left.\frac{d}{dt}\right|_{t=0} f \circ c_p(t) = X_p(f). \quad \square$$

Now consider Vect(X) as a vector space and the action of Diff(M) on Vect(M) given by $(\varphi, Y) \mapsto \varphi^* Y = (\varphi^{-1})_* Y$. We have $D\varphi^{-1}\big|_{\varphi(p)} : T_{\varphi(p)M} \to T_p M$ so that

$$(\varphi^* Y)\big|_p (g) = D\varphi^{-1}\big|_{\varphi(p)} (Y)(g) = Y_{\varphi(p)}(g \circ \varphi^{-1}) \tag{1}$$

Definition 5.12. The Lie derivative of $Y \in \text{Vect}(M)$ is

$$L_X Y = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* Y.$$

Proposition 5.13. $L_X Y = [X, Y]$.

Proof. Put $g = f \circ \varphi$ in (1):

$$\varphi^* Y\big|_p (f \circ \varphi) = Y_{\varphi(p)}(f) =: h(\varphi(p)),$$

i.e. set $h = Y(f)$. We have

$$\begin{aligned} L_X Y\big|_p (f) &= \lim_{t \to 0} \frac{\varphi_t^* Y\big|_p (f) - Y_p(f)}{t} \\ &= \underbrace{\lim_{t \to 0} \frac{\varphi_t^* Y\big|_p (f) - \varphi_t^* Y\big|_p (f \circ \varphi_t)}{t}}_{=:A} + \underbrace{\lim_{t \to 0} \frac{\varphi_t^* Y\big|_p (f \circ \varphi_t) - Y_p(f)}{t}}_{=:B}. \end{aligned}$$

By linearity,

$$A = \lim_{t \to 0} \overbrace{\varphi_t^* Y\big|_p}^{\to Y|_p} \underbrace{\left(\frac{f - f \circ \varphi_t}{t} \right)}_{\to -\left.\frac{d}{dt}\right|_{t=0} \varphi_t^*(f) = -X(f)} = -\,Y\big|_p (X(f)).$$

$$B = \lim_{t \to 0} \frac{h(\varphi_t(p)) - h(p)}{t} = L_X h\big|_p = X_p(h) = X_p(Y(f)).$$

So $L_X Y = XY - YX$. ☐

Corollary 5.14.

1. *$L_X Y$ is linear with respect to both X and Y.*
2. *$L_X Y = -L_Y X$.*
3. *$L_X X = 0$. If $X_p = 0$ and $Y_p = 0$, then $L_X Y\big|_p = 0$.*

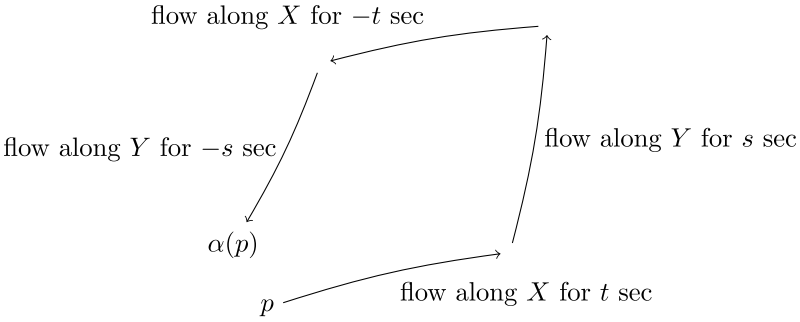
Lemma 5.15.

1. *Let $\alpha \colon M \to N$ be a diffeomorphism and $X \in \text{Vect}(M)$ with flow φ_t. Then the flow of $\alpha_* X$ is $\alpha \circ \varphi_t \circ \alpha^{-1}$.*
2. *Let $\alpha \colon M \to M$ be a diffeomorphism and $X \in \text{Vect}(M)$ with flow φ_t. Then $\alpha_* X = X$ if and only if $\alpha \circ \varphi_t = \varphi_t \circ \alpha$.*

Proof. exercise. ☐

Proposition 5.16. *Let X and Y be vector fields on M with flows φ_t and ξ_s respectively. Then $[X, Y] = 0$ if and only if $\varphi_t \circ \xi_s = \xi_s \circ \varphi_t$ for all s, t.*

Consider $\alpha = \xi_{-s} \circ \varphi_{-t} \circ \xi_s \circ \varphi_t$. Then the proposition says that $\alpha(p) = p$ for all p and all s, t if and only if $[X, Y] = 0$.



Proof. If φ and ξ commute then by the second point of the lemma, $\varphi_t^* Y = Y$ and hence $[X, Y] = L_X Y = 0$.

Suppose now that $[X, Y] = 0$. So $L_X Y = \lim_{h\to 0} \frac{Y_q - (\varphi_h)_* Y_q}{h} = 0$. Fix $p \in M$ and set $c(t) = (\varphi_t)_* \left. Y\right|_p$. Then

$$\begin{aligned} c'(t) &= \lim_{h\to 0} \frac{1}{h}(c(t+h) - c(t)) \\ &= \lim_{h\to 0} \frac{1}{h}\left((\varphi_{t+h})_* \left. Y\right|_p - (\varphi_t)_* \left. Y\right|_p \right) \\ &= \lim_{h\to 0} (\varphi_t)_* \underbrace{\left(\frac{(\varphi_h)_* \left. Y\right|_p - \left. Y\right|_p}{h} \right)}_{\to 0} \\ &= (\varphi_t)_* (0) = 0. \end{aligned}$$

Thus c is constant, i.e. $c(t) = c(0)$ for all t. So $(\varphi_t)_* \left. Y\right|_p = \left. Y\right|_p$. This is true for all p, so that $(\varphi_t)_* Y = Y$ for all t.

By the lemma this implies that φ_t and ξ_s commute. □

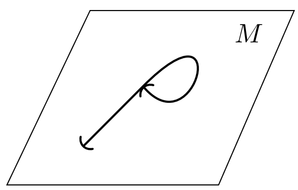
6 Submanifolds

Definition 6.1. A *submanifold* of a manifold M is a manifold N together with a smooth injection $\iota \colon N \to M$ such that

1. for all $p \in N$, $\left. D\iota\right|_p : T_p N \to T_{\iota(p)} M$ is injective; and
2. $\iota(N)$ has the subspace topology.

If only 1 holds, then we say that N is an *immersed submanifold* and ι an immersion. A submanifold is sometimes called an *embedded submanifold*.

Example 6.2.



is an immersion, but not a submanifold.

Theorem 6.3. *Let $F\colon M \to N$ be smooth. Let $c \in N$ and set $Z = F^{-1}(c)$. Suppose* $\operatorname{rank}\left(DF|_p \right) = \dim N$ *for all $p \in Z$. Then Z is a submanifold of M (or $Z = \emptyset$) of dimension* $\dim Z = \dim M - \dim N$. *If $\iota Z \hookrightarrow M$ is the inclusion, then $D\iota|_p : T_pZ \xrightarrow{\sim} \ker(DF|_p : T_pM \to T_{F(p)}N)$.*

Theorem 6.4 (Inverse Function Theorem)**.** *Let $U \subseteq_o \mathbb{R}^N$ and $G\colon U \to \mathbb{R}^N$ smooth such that $DG|_p$ is an isomorphism at some point $p \in U$. Then G is a local diffeomorphism around p (i.e. there exist neighborhoods $p \in V \subseteq_o U$ and $G(p) \in W$ such that G induces a diffeomorphism $G\colon V \to W$).*

Proof. handout. □

Proof of the submanifold theorem. Let $n = \dim N$, $\dim M = n + m$. Fix $p \in Z$ and local coordinates $x^1, \ldots, x^{m+n}$ on U around p and $y^1, \ldots, y^n$ around $F(p) = c$. Say $F = (F_1, \ldots, F_n)$, $DF|_p = \left(\frac{\partial F_i}{\partial x^j}\Big|_p \right)_{1\le i\le n,\, 1\le j\le m+n}$ and $c = (c_1, \ldots, c_n)$ with respect to these coordinates.

As $DF|_p$ is a surjection, it has a non-singular $n \times n$ minor. Without loss of generality assume that $A = \left(\frac{\partial F_i}{\partial x^j}\Big|_p \right)_{1\le i,j\le n}$ is non-singular. Set

$$G\colon U \to \mathbb{R}^{n+m}, \qquad (x_1, \ldots, x_{n+m}) \mapsto (F_1(x_1, \ldots, x_{n+m}), \ldots, F_n(x_1, \ldots, x_{n+m}), x_{n+1}, \ldots, x_{n+m}).$$

Then $DG|_p = \left(\begin{smallmatrix} A & * \\ 0 & I \end{smallmatrix}\right)$ is an isomorphism, so by the inverse function theorem there are neighborhoods $p \in U' \subseteq U$, $G(p) \in V'$ such that $G|_{U'} : U' \xrightarrow{\sim} V'$.

Consider $\widetilde{U} = U' \cap Z$. Then $G|_{\widetilde{U}}$ has image $\{(c_1, \ldots, c_n, x_{n+1}, \ldots, x_{n+m})\} \cong \{(x_{n+1}, \ldots, x_{n+m})\} \subseteq \mathbb{R}^m$. This gives a chart on Z and the transition functions are smooth since they are given by a composition of smooth functions. □

Example 6.5. $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $(x_0, \ldots, x_n) \mapsto x_0^2 + \cdots x_n^2$ gives $S^n = F^{-1}(1)$. For a fixed $p \in S^n$, $DF|_p(v_0, \ldots, v_n) = 2\sum_i p_i v_i$. So $DF|_p$ is onto (i.e. has rank 1) for all $p \in S^n$. Thus S^n is a submanifold of $\mathbb{R}^{n+1}$ and

$$T_pS^n = \{(v_0, \ldots, v_n) : DF|_p(v_0, \ldots, v_n) = 0\} = \{\mathsf{v} : \mathsf{v} \cdot \mathsf{p} = 0\}.$$

Example 6.6. The symmetric matrices $\mathrm{Sym}_n = \{A \in M_n : A = A^T\}$ are a vector space (and thus a manifold) of dimension $\frac{n(n+1)}{2}$. Consider $F\colon \mathrm{GL}_n \to \mathrm{Sym}_n$, $A \mapsto AA^T$. So $O(n) = F^{-1}(I)$.

For $B \in O(n)$, the map $R_B\colon \mathrm{GL}_n \to \mathrm{GL}_n$, $A \mapsto AB$ is a diffeomorphism and $F \circ R_B(A) = ABB^TA^T = AA^T = F(A)$, i.e. $F \circ R_B = F$. Thus $DF|_I = DF|_B \circ DR_B|_I$. Hence $DF|_B$ is surjective if and only if $DF|_I$ is surjective. From $F(I + tH) = I + t(H + H^T) + O(t^2)$, we see that $DF|_I(H) = H + H^T$ is surjective.

Therefore $O(n)$ is a manifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Definition 6.7. Let $\iota\colon N \to M$ be a smooth embedding. Two vector fields $X \in \mathrm{Vect}(N)$ and $Y \in \mathrm{Vect}(M)$ are called *ι_*-related*, if $D\iota|_p(X_p) = Y_{\iota(p)}$ for all $p \in N$.

The condition is equivalent to $Y_{\iota(p)}(g) = D\iota|_p(X_p)(g) = X_p(g \circ \iota)$ for all p and g and hence to $Y(g) \circ \iota = X(g \circ \iota)$.

Proposition 6.8. *If $X_j \in \mathrm{Vect}(N)$ are ι_*-related to $Y_j \in \mathrm{Vect}(M)$ for $j = 1, 2$, then $[X_1, X_2]$ is ι_*-related to $[Y_1, Y_2]$.*

Proof.

$$\begin{aligned} [Y_1, Y_2](g) \circ \iota &= (Y_1(Y_2(g)) - Y_2(Y_1(g))) \circ \iota \\ &= X_1(Y_2(g) \circ \iota) - X_2(Y_1(g) \circ \iota) \\ &= X_1(X_2(g \circ \iota)) - X_2(X_1(g \circ \iota)) = [X_1, X_2](G \circ \iota). \end{aligned}$$

□

Example 6.9. A *Lie subgroup* of a Lie group G is a subgroup $H \subseteq G$, that is an immersed submanifold. Say $\iota\colon H \to G$. Let $\mathfrak{h}$ and $\mathfrak{g}$ be the corresponding Lie algebras. Then $D\iota = \iota_*\colon \mathfrak{h} \to \mathfrak{g}$. We will show that ι_* is a map of Lie algebras, so that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

Let $L_a\colon H \to H$ and $\widetilde{L}_a\colon G \to G$ be left multiplication. Then $\widetilde{L}_a \circ \iota = \iota \circ L_a$. Fix $\xi \in \mathfrak{h}$ which gives us $X_\xi \in \mathrm{Vect}_L(H)$. We have to show that $\iota_* X_\xi \in \mathrm{Vect}_L(G)$:

$$\left(\widetilde{L}_a\right)_* \iota_* X_\xi = \iota_*(L_a)_* X_\xi = \iota_* X_\xi.$$

Also for $\eta \in \mathfrak{h}$, $[X_\xi, X_\eta]$ is ι_* related to $[\iota_* X_\xi, \iota_* X_\eta]$ so that ι_* preserves the Lie bracket.

In fact, the converse is also true: Given a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, the exists a Lie subgroup $H \subseteq G$ with Lie algebra $\mathfrak{h}$.

7 Integral manifolds

Definition 7.1.

1. A k-dimensional *distribution* Δ on M is a choice of k-dimensional subspace $\Delta_p \subseteq T_pM$ for each $p \in M$.

2. Say Δ is smooth if for all $p \in M$ there exists an open $p \in U$ and $X_1, \ldots, X_k \in \mathrm{Vect}(U)$ such that $\mathrm{span}_{\mathbb{R}}\{ X_1|_q, \ldots, X_k|_q\} = \Delta_q$ for all $q \in U$.

3. An immersion $\iota\colon N \to M$ is an *integral manifold* of Δ, if $\iota_* T_qN = \Delta_q$ for all $q \in N$.

Suppose that Δ is a distribution on M and $X_1, X_2 \in \mathrm{Vect}(M)$ such that $X_1|_p, X_2|_p \in \Delta_p$. Suppose further that there exists an integral manifold $\iota\colon N \hookrightarrow M$ of Δ with $p \in N$. Define $\widetilde{X_j}\Big|_q$ by $\iota_*(\widetilde{X_j}\Big|_q) = X_j|_q$ for all $q \in N$, $j = 1, 2$. (This is possible as $\iota_*(T_qN) = \Delta_q$ and ι_* is injective.) Then $X_j \in \mathrm{Vect}(N)$ (exercise). By definition, $\widetilde{X_j}$ is ι_*-related to X_j, so that $[\widetilde{X_1}, \widetilde{X_2}]$ is ι_*-related to $[X_1, X_2]$. Thus $[X_1, X_2]_p \in \Delta_p$. This motivates the following definition:

Definition 7.2. Let Δ be a smooth distribution on M.

1. Say $X \in \mathrm{Vect}(M)$ *belongs* to Δ if $X_p \in \Delta_p$ for all $p \in M$.

2. Say Δ is *integral* (or *integrable* or *involutive*) if whenever X, Y belong to Δ, then $[X, Y]$ belongs to Δ.

Theorem 7.3 (Frobenius Integrability Theorem)**.** *If Δ is integral if and only if there exists an integral manifold of Δ through every point of M.*

Lemma 7.4. *Suppose $X_1, \ldots, X_k \in \mathrm{Vect}(U)$ are linearly independent and $[X_i, X_j] = 0$ for all i, j. Let $p \in M$. Then there exist coordinates $y^1, \ldots, y^n$ around p such that $X_i = \frac{\partial}{\partial y^i}$ for $i = 1, \ldots, k$ near p.*

Proof. by working locally, we may assume $M = \mathbb{R}^n$, $p = 0$. Further, by a linear change of coordinates we may assume $X_i|_0 = \frac{\partial}{\partial x^i}\big|_0$ for $i = 1, \ldots, k$.

Say X_i has flow $\varphi_i^{(t)}$. We will construct the first k coordinates using these flows. Consider

$$F\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad F(t_1, \ldots, t_n) = \varphi_{t_1}^{(1)} \varphi_{t_2}^{(2)} \cdots \varphi_{t_k}^{(k)}(0, \ldots, 0, t_{k+1}, \ldots, t_n).$$

Using the inverse function theorem we want to show that F is a local diffeomorphism. For this we show that $DF|_0 = I$.

Fix $\mathsf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} DF\left(\left.\frac{\partial}{\partial x^1}\right|_{\mathsf{a}} \right)(g) &= \left.\frac{\partial}{\partial t_1}\right|_{\mathsf{a}} (g \circ F) \\ &= \lim_{h\to 0} \frac{g(F(a_1+h, a_2, \dots, a_n)) - g(F(\mathsf{a}))}{h} \\ &= \lim_{h\to 0} \frac{g(\varphi^{(1)}_{a_1+h}\varphi^{(2)}_{a_2}\cdots\varphi^{(k)}_{a_k}(0,\dots,0,a_{k+1},\dots,a_n)) - g(F(\mathsf{a}))}{h} \\ &= \lim_{h\to 0} \frac{g(\varphi^{(1)}_h(F(\mathsf{a}))) - g(F(\mathsf{a}))}{h} \\ &= L_{X_1} g(F(\mathsf{a})) = X_1(g)|_{F(\mathsf{a})}. \end{aligned}$$

Thus

$$\begin{aligned} DF\left(\left.\frac{\partial}{\partial x^1}\right|_{\mathsf{a}} \right) &= X_1|_{F(\mathsf{a})}, \\ DF\left(\left.\frac{\partial}{\partial x^1}\right|_{\mathsf{0}} \right) &= X_1|_{F(\mathsf{0})} = X_1|_{\mathsf{0}} = \left.\frac{\partial}{\partial x^1}\right|_{\mathsf{0}}. \end{aligned}$$

Since $[X_i, X_j] = 0$, the flows commute, so that the same calculation works to show

$$DF\left(\left.\frac{\partial}{\partial x^i}\right|_{\mathsf{a}} \right) = X_i|_{F(\mathsf{a})}, \tag{2}$$

$$DF\left(\left.\frac{\partial}{\partial x^i}\right|_{\mathsf{0}} \right) = X_i|_{\mathsf{0}} = \left.\frac{\partial}{\partial x^i}\right|_{\mathsf{0}} \tag{3}$$

for $i = 1, \dots, k$. For $i > k$ we have

$$\begin{aligned} DF\left(\left.\frac{\partial}{\partial x^i}\right|_{\mathsf{0}} \right)(g) &= \left.\frac{\partial}{\partial t_i}\right|_{\mathsf{0}} (g \circ F) \\ &= \lim_{h\to 0} \frac{g(F(0,\dots,0,h,0,\dots,0)) - g(F(\mathsf{0}))}{h} \\ &= \lim_{h\to 0} \frac{g(0,\dots,0,h,0,\dots,0) - g(\mathsf{0})}{h} \\ &= \left.\frac{\partial g}{\partial x_i}\right|_{\mathsf{0}}. \end{aligned}$$

Thus $DF|_{\mathsf{0}} = I$ and by the inverse function theorem F is a local diffeomorphism. Hence $y = F^{-1}$ is a coordinate system and by (2), $\frac{\partial}{\partial y^i} = X_i$ for $i = 1, \dots, k$. □

Proof of Frobenius' Theorem. We may assume that we are working on $\mathbb{R}^n$ with $p = 0$ and $\Delta_0 = \operatorname{span}_{\mathbb{R}}\{ \left.\frac{\partial}{\partial x_1}\right|_{\mathsf{0}}, \dots, \left.\frac{\partial}{\partial x_k}\right|_{\mathsf{0}} \}$. Consider

$$\pi\colon \mathbb{R}^n \to \mathbb{R}^k, \qquad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_k).$$

So $D\pi|_{\mathsf{0}} : \Delta \to T_0\mathbb{R}^k$ is an isomorphism. So for q near 0, $D\pi|_q : \Delta_q \to T_{\pi(q)}\mathbb{R}^k$ is an isomorphism and in particular injective. Thus there exist $X_1, \dots, X_k \in \Delta$ such that $\pi_* X_j(q) = \left.\frac{\partial}{\partial x_j}\right|_{\pi(q)}$ for q near 0 and $j = 1, \dots, k$.

By assumption $[X_i, X_j] \in \Delta$. So $\pi_*[X_i, X_j] = [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$. By the lemma there exist coordinates $y^1, \dots, y^n$ with $X_i = \frac{\partial}{\partial y^i}$ for $i = 1, \dots, k$. So $N = \{y^{k+1} = \cdots = y^n = 0\}$ is an integral manifold of

Δ near $p = 0$. □

Remark 7.5. With a little more work one can show that there exists a unique maximal integral manifold through p.

Example 7.6. Let G be a Lie group and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. We will show that there exists a Lie group $H \subseteq G$ with Lie group $\mathfrak{h}$.

For $g \in G$ set $\Delta_g = \operatorname{span}_{\mathbb{R}}\{\, X_\xi\big|_g : \xi \in \mathfrak{h}\}$. This is a smooth distribution and as $\mathfrak{h}$ is closed under the Lie bracket, Δ is integrable. So by the Frobenius theorem there exists a maximal integral manifold H through $e \in G$.

For $g \in G$ note $(L_g)_*\Delta_{g'} = (L_{gg'})_*\Delta_e$, so $(L_g)_*$ takes Δ to itself. One can check that this implies that L_g takes the integral manifold through e to that through g. If $h \in H$, then $L_{h^{-1}}$ takes H to the unique maximal integral manifold through $L_{h^{-1}}(h) = e$, i.e. $L_{h^{-1}}$ takes H to H so that H is closed under multiplication. So it is a subgroup. One can show that multiplication is smooth on H.

8 Vector bundles

The idea of a vector bundle is to give a vector space on each point of M, varying smoothly.

Definition 8.1. A *(smooth) vector bundle* E of *rank* r on a manifold M consists of a smooth manifold E and a smooth map $\pi\colon E \to M$ such that

1. for all $p \in M$, $E_p = \pi^{-1}(p)$ is an r-dimensional real vector space;
2. for all $p \in M$ there exists an open neighborhood U and a diffeomorphism $t_U\colon\ E|_U \to U \times \mathbb{R}^n$ (where $E|_U = \pi^{-1}(U)$) such that
   a) $t_U|_{E_q} : E_q \to \{q\} \times \mathbb{R}^n$ is a linear isomorphism and
   b) The diagram

$$\begin{array}{ccc} E|_U & \xrightarrow{\ t_U\ } & U \times \mathbb{R}^r \\ & {\scriptstyle \pi|_U} \searrow \quad \swarrow {\scriptstyle p} & \\ & U & \end{array}$$

   commutes.

E is called the *total space*, M the *base space*, E_p is the *fiber* of E over p and t_U is a *trivialization* of E over U. If $r = 1$, then E is called a *line bundle*.

Example 8.2. The tangent bundle TM is a vector bundle: Given a coordinate chart $x^1, \dots, x^n$, then $\frac{\partial}{\partial x^1}\big|_q, \dots, \frac{\partial}{\partial x^n}\big|_q$ are a basis for T_qM. Set

$$t_U\colon\ TM|_U = TU \to U \times \mathbb{R}^n, \qquad v_q \mapsto (q, (a^1, \dots, a^n)),$$

where $v_q = a^1\ \frac{\partial}{\partial x^1}\big|_q + \cdots a^n\ \frac{\partial}{\partial x^n}\big|_q \in T_qM$.

Example 8.3. $E = M \otimes \mathbb{R}^n$ is called the *trivial bundle*.

Suppose that t_U and t_V are trivializations of E and consider

$$t_U \circ t_V^{-1}\colon (U \cap V) \times \mathbb{R}^r \to (U \cap V) \times \mathbb{R}^r, \qquad (p, w) \mapsto (p, w').$$

The map $w \mapsto w'$ is a linear isomorphism, i.e. $t_U \circ t_V^{-1}(p, w) = (p, \varphi_{UV}(p)(w))$ with $\varphi_{UV}\colon U \cap V \to \mathrm{GL}_r(\mathbb{R})$. The map φ_{UV} is called the *transition function*.

Remark 8.4. $t_U \circ t_V^{-1}$ is smooth if and only if φ_{UV} is smooth.

Observe:

1. $\varphi_{UU} = I$ (as $t_U \circ t_U^{-1} = \mathrm{id}$);

2. $\varphi_{UV} = \varphi_{VU}^{-1}$ (as $(t_U \circ t_V^{-1})^{-1} = t_V \circ t_U^{-1}$);

3. $\varphi_{UV}\varphi_{VW} = \varphi_{UW}$ (as $t_U t_V^{-1} t_V t_W^{-1} = t_U t_W^{-1}$).

Conditions 1–3 are called the *cocycle condition.* Given an open cover $\{U_\alpha\}$ and functions $\{\varphi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{R})\}$, call $\{\varphi_{\alpha\beta}\}$ a *cocycle* if $\varphi_{\alpha\alpha} = I$, $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$ and $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$.

8.1 Vector bundle constructions

Proposition 8.5. *Suppose for every point $p \in M$ we have a vector space E_p of dimension r. Set $E = \coprod_{p\in M} E_p$ and $\varphi \colon E \to M$ sending $v_p \in E_p$ to p. Suppose further that $\{U_\alpha\}$ is an open cover by charts and that for all α we have a bijective map $t_{U_\alpha} \colon E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r$ such that*

1. *The diagram*

$$\begin{array}{ccc} E|_{U_\alpha} & \xrightarrow{t_{U_\alpha}} & U_\alpha \times \mathbb{R}^r \\ & {\scriptstyle \pi|_{U_\alpha}} \searrow \quad \swarrow {\scriptstyle p} & \\ & U_\alpha & \end{array}$$

   commutes;

2. *$t_{U_\alpha} \colon E_q \cong \{q\} \times \mathbb{R}^n$ for all $q \in U_\alpha$; and*

3. *$t_{U_\alpha} t_{U_\beta}^{-1}(p, v) = (p, \varphi_{\alpha\beta}(p)v)$ with $\varphi_{\alpha\beta}$ smooth.*

Then there exists a unique manifold structure on E making it into a vector bundle over M with a cocycle $\{\varphi_{\alpha\beta}\}$.

Proof. Suppose our charts are (x_α, U_α). Charts on E are given by

$$E|_{U_\alpha} \xrightarrow{t_{U_\alpha}} U_\alpha \times \mathbb{R}^r \xrightarrow{x_\alpha} x_\alpha(U_\alpha) \times \mathbb{R}^r \cong \mathbb{R}^{n+r}.$$

Exercise: The transition functions are smooth. The induced topology is Hausdorff and second countable. □

Example 8.6. Let E and F be vector bundles on M. Define $E\oplus F$ as follows: Set $E\oplus F = \coprod_{p\in M} E_p \oplus F_p$. We can choose an open cover of M by U_α such that there are trivialization t_{U_α} and s_{U_α} of E and F. Consider $E \oplus F|_{U_\alpha} \to U_\alpha \times \mathbb{R}^{r+r'}$ given by these functions. One can check that if E has transition functions $\varphi_{\alpha\beta}$ and F has transition functions $\xi_{\alpha\beta}$, then $E \oplus F$ has transition functions $\varphi_{\alpha\beta} \oplus \xi_{\alpha\beta} = \left(\begin{smallmatrix} \varphi_{\alpha\beta} & 0 \\ 0 & \xi_{\alpha\beta} \end{smallmatrix}\right)$, which is smooth.

Example 8.7. Let $M = \mathbb{RP}^n$. Define a line bundle L on M by

$$L_{[\mathsf{v}]} = \text{line in } \mathbb{R}^{n+1} \text{ through } \mathsf{v} = \{\lambda\mathsf{v} : \lambda \in \mathbb{R}\}.$$

L is called the *Hopf line bundle* or the *tautological bundle.* It is indeed a vector bundle:

$$L = \coprod_{[\mathsf{v}]\in\mathbb{RP}^n} L_{[\mathsf{v}]}.$$

Consider the charts $U_i = \{v_i \neq 0\} \subseteq \mathbb{RP}^n$, $U_i \cong \mathbb{R}^n$, and set

$$t_{U_i} \colon L|_{U_i} \to U_i \times \mathbb{R}, \qquad \lambda(v_0, \ldots, v_n) \mapsto ([v_0 : \cdots : v_n], \lambda v_i)\,.$$

This is linear on $L_{[\mathsf{v}]}$, so gives a trivialization. We have

$$t_{U_i} t_{U_j}^{-1} \colon (U_i \cap U_j) \times \mathbb{R} \to (U_i \cap U_j) \times \mathbb{R}, \qquad ([\mathsf{v}], \lambda) \mapsto ([\mathsf{v}], \frac{v_i}{v_j} \lambda).$$

So $\varphi_{U_i U_j} = \frac{v_i}{v_j} \in \mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^*$ is smooth.

Example 8.8 (Dual bundle). Suppose $E \to M$ is a vector bundle of rank r and set $E^* = \coprod_{p \in M} E_p^*$. To make E^* into a bundle, take a trivialization $t_U \colon E|_U \to U \times \mathbb{R}^r$, say $t_U(v_p) = (p, F_p(v_p))$ with $F_p \colon E_p \to \mathbb{R}^r$ linear. Then $\left(F_p^{-1}\right)^* \colon E_p^* \to (\mathbb{R}^r)^* \cong \mathbb{R}^r$. So consider

$$\widetilde{t}_U \colon E^*|_U \to U \times \mathbb{R}^r, \qquad \alpha \in E_p \mapsto \left(p, \left(F_p^{-1}\right)^* (\alpha_p)\right).$$

If E has smooth transition functions $\varphi_{UV} \colon U \cap V \to \mathrm{GL}_r$, then E^* has transition functions $\left(\varphi_{UV}^{-1}\right)^T$, which are also smooth. So E^* is a vector bundle, called the *dual* of E.

Definition 8.9. The *cotangent bundle* to M is $T^*M = (TM)^*$.

Remark 8.10. Say we have coordinates $x^1, \dots, x^n$ on U. There are maps

$$U \to TM|_U, \qquad p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p.$$

There are also maps

$$U \to T^*M|_U, \qquad p \mapsto dx^i\big|_p.$$

Both of them are smooth (exercise). Recall: $\{dx^i\}$ are dual to $\{\frac{\partial}{\partial x^i}\}$, i.e. $dx^i\big|_p \left(\frac{\partial}{\partial x^j}\big|_p\right) = \delta_{ij}$.

Definition 8.11. Given a bundle $E \xrightarrow{\pi} M$, a *section* over $U \subseteq_o M$ is a smooth map $s \colon U \to E$ such that $\pi \circ s = \mathrm{id}_U$. The space of sections of E over U is denoted $C^\infty(U, E)$ or $\Gamma(U, E)$ and is a $C^\infty(M)$-module.

Example 8.12. $C^\infty(M, TM) = \mathrm{Vect}(M)$ and $C^\infty(M, M \times \mathbb{R}) = C^\infty(M)$.

Definition 8.13. $\Omega^1(M) = C^\infty(M, T^*M)$ are the 1-*forms* of M.

Remark 8.14. Observe the pairing

$$\Omega^1(M) \times \mathrm{Vect}(M) \to C^\infty(M), \qquad (\omega, X) \mapsto (p \mapsto \omega_p(X_p)).$$

If $f \in C^\infty(M)$, then $Df|_p \colon T_pM \to \mathbb{R}$ is an element of T_p^*M. So we get $df \in \Omega^1(M)$, defined by $df|_p (X_p) = X(f)|_p$. In coordinates this is $df = \sum \frac{\partial f}{\partial x^i} dx^i$.

Definition 8.15. Let $\varphi \colon E \to M$ and $\pi' \colon E' \to M'$ be vector bundles. A *bundle map* from E to E' consists of a pair (F, f), where $f \colon M \to M'$ and $F \colon E \to E'$ are smooth such that

1. The diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow \pi & & \downarrow \pi' \\ M & \xrightarrow{f} & M' \end{array}$$

   commutes.

2. $F \colon E_p \to E'_{f(p)}$ is linear for all $p \in M$.

Example 8.16. If $f\colon M \to M'$ is smooth, then $Df\colon TM \to TM'$ forms a bundle map.

Definition 8.17. Given $E \to M$ and $E' \to M$, a bundle map *over* M is a bundle map (F, id_M). A bundle isomorphism is a bundle map with an inverse that is also a bundle map.

Remark 8.18. Given a bundle map $F\colon E \to E'$ over M and $U \subseteq_o M$ we get an induced map

$$\alpha\colon C^\infty(M,E) \to C^\infty(M,E'), \qquad s \mapsto F \circ s,$$

i.e. $\alpha(s)(p) = F(s(p))$. The map α is $C^\infty(U)$-linear: For $g \in C^\infty(U)$ we have

$$\alpha(gs)(p) = F((gs)(p)) = F(g(p)s(p)) = g(p)F(s(p)),$$

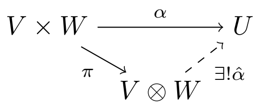
i.e. $\alpha(gs) = g\alpha(s)$.

Proposition 8.19. *Suppose that* $\alpha\colon C^\infty(M,E) \to C^\infty(M,E')$ *is* $C^\infty(M)$*-linear. Then there exists a unique bundle map (over* M*)* $F\colon E \to E'$ *such that* $\alpha(s) = F \circ s$.

Proof. Example sheet. □

9 Tensors and exterior algebra

Definition 9.1. Let V and W be vector spaces. The *tensor product* of V and W is a vector space $V \otimes W$ and a bilinear map $\pi\colon V \times W \to V \otimes W$ such that if $\alpha\colon V \times W \to U$ is a bilinear map to some vector space U, then there exists a unique linear map $\hat{\alpha}\colon V \otimes W \to U$ making the diagram



commute.

Assume that the spaces are finite dimensional. For $U = \mathbb{R}$ the diagram says that $(V \otimes W)^* = \mathrm{Bilinear}(V \times W, \mathbb{R})$. Thus we can set $V \otimes W = \mathrm{Bilinear}(V \times W, \mathbb{R})^*$, which shows existence of the tensor product.

For $v \in V$ and $w \in W$ we write $v \otimes w$ for $\pi(v,w)$. Note that π is in general not surjective!

Proposition 9.2. *The tensor product is functorial, i.e. if* $f\colon V \to V'$ *and* $g\colon W \to W'$ *are linear, then there exists a unique linear map* $f \otimes g\colon V \otimes W \to V' \otimes W'$ *such that*

$$\begin{array}{ccc} V \times W & \xrightarrow{f \times g} & V' \times W' \\ \downarrow & & \downarrow \\ V \otimes W & \xrightarrow{f \otimes g} & V' \otimes W' \end{array}$$

commutes. The following isomorphisms hold and are natural, i.e. compatible with the diagram above:

1. $V \otimes W \cong W \otimes V$;
2. $V \otimes (U \otimes W) \cong (V \otimes U) \otimes W$;
3. $(V \otimes W)^* \cong V^* \otimes W^*$.

Lemma 9.3. *If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are bases for V and W, then*

$$\{v_i \otimes w_j : i = 1, \ldots, n,\ j = 1, \ldots, m\}$$

is a basis for $V \otimes W$.

Remark 9.4. Given $\omega \in V^* \otimes W^*$, say $\omega = \sum_{i,j} \alpha_i \otimes \beta_j$, we can think of ω as a bilinear map

$$\omega \colon V \times W \to \mathbb{R}, \qquad \omega(v, w) = \sum_{i,j} \alpha_i(v)\beta_j(w).$$

Remark 9.5. Using the vector bundle construction, if $E \to M$ and $F \to M$ are vector bundles, we have a bundle $E \otimes F$ whose fibers are $(E \otimes F)_p = E_p \otimes F_p$ (exercise).

Definition 9.6. Set $T^{(k,l)}M = (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$. The elements of $\mathcal{T}^{(k,l)}(U) = C^\infty(U, T^{(k,l)}U)$ are called *mixed tensors.*

Example 9.7. $\mathcal{T}^{(1,0)}(U) = \mathrm{Vect}(U)$ and $\mathcal{T}^{(0,1)}(U) = \Omega^1(U)$.

Remark 9.8. Given coordinates $x^1, \ldots, x^n$ on U and $\omega \in \mathcal{T}^{(k,l)}(U)$, we can write

$$\omega = \sum \alpha^{i_1,\ldots,i_k}_{j_1,\ldots,j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}$$

with $\alpha^{i_1,\ldots,i_k}_{j_1,\ldots,j_l} \in C^\infty(U)$. There is a change of coordinates formula similar to that for $\frac{\partial}{\partial x^i}$ and dx^i.

Remark 9.9. $V \otimes W^* \cong \mathrm{Hom}(W, V)$ via $\varphi \colon V \otimes W^* \to \mathrm{Hom}(W, V)$ given by $\varphi(v \otimes \alpha)(w) = \alpha(w)v$ and linear extension.

Remark 9.10. Say $\omega \in V \otimes W \otimes W^* \otimes U$. Then we can form the *contraction* of ω, $C(\omega) \in V \otimes U$ by $C(v \otimes w \otimes \alpha \otimes u) = \alpha(w)v \otimes u$ and linear extension.

Example 9.11. Let $\omega \in \mathcal{T}^{(k,l)}(U)$ be written in local coordinates as above. Then we can form the *a-b-contraction*

$$C^a_b(\omega) = \sum \alpha^{\mathrm{i}}_{\mathrm{j}} \left(\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \widehat{\frac{\partial}{\partial x^{i_a}}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes \widehat{dx^{j_b}} \otimes \cdots \otimes dx^{j_l} \right) \cdot \underbrace{dx^{j_b}\left(\frac{\partial}{\partial x^{i_a}}\right)}_{=\delta_{i_a, j_b}}.$$

Definition 9.12. Set $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ (with $V^{\otimes 0} = \mathbb{R}$). The maps

$$V^{\otimes k} \times V^{\otimes l} \to V^{\otimes (k+l)}, \qquad (v_1, v_2) \mapsto v_1 \otimes v_2$$

make $T(V)$ into an algebra, called the *tensor algebra* of V.

Definition 9.13. Let I be the two-sided ideal of $T(V)$ generated by elements $v \otimes v$, $v \in V$. The *exterior algebra* of V is $\Lambda V = T(V)/I$. Let $\pi \colon T(V) \to \Lambda V$ be the projection. Then $\Lambda^r V = \pi(V^{\otimes r})$ is the *r-th exterior power* of V.

Notation. The multiplication on $T(V)$ gives a multiplication on ΛV, denoted $\wedge$.

Remark 9.14. For $v_1, v_2 \in V$: $v_1 \wedge v_2 = -v_2 \wedge v_1$.

The idea is to consider $\Lambda^r V^*$ as the space $\mathrm{Alt}^r(V)$ of alternating forms $V \times \cdots \times V \to \mathbb{R}$. Say $\alpha_1 \wedge \cdots \wedge \alpha_r \in \Lambda^r V^*$ for $\alpha_i \in V^*$. Then

$$(\alpha_1 \wedge \cdots \wedge \alpha_r)(v_1, \ldots, v_r) = \sum_{\pi \in S_r} \varepsilon(\pi) \alpha_{\pi(1)}(v_1) \cdots \alpha_{\pi(r)}(v_r),$$

where $\varepsilon(\pi)$ is the sign of the permutation π.

Lemma 9.15.

1. *If* $\alpha \in \Lambda^p V$, $\beta \in \Lambda^q V$, *then* $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.
2. $\dim \Lambda^0 V = 1$; $\Lambda^1 V = V$, $\dim \Lambda^n V = 1$, *where* $n = \dim V$.
3. *If* $v_1, \dots, v_n$ *is a basis for* V, *then* $B = \{v_{i_1} \wedge \cdots \wedge v_{i_r} : i_1 < \cdots < i_r\}$ *is a basis for* $\Lambda^r V$.

Proof.

1. If $v \in V$, then $v \wedge v = 0$, so for $v_1, v_2 \in V$: $0 = (v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_2 + v_2 \wedge v_1$. So $v_1 \wedge v_2 = -v_2 \wedge v_1$.

   Hence swapping two adjacent elements in $v_1 \wedge \cdots \wedge v_r$ changes the sign. Thus

   $$v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q = (-1)^{pq} w_1 \wedge \cdots \wedge w_q \wedge v_1 \wedge \cdots \wedge v_p$$

   and clearly any element $\alpha \in \Lambda^p V$ (resp. $\beta \in \Lambda^q V$) is a linear sum of vectors of the form $v_1 \wedge \cdots \wedge v_p$ (resp. $w_1 \wedge \cdots \wedge w_q$).

2. $\Lambda^0 V = \pi(V^{\otimes 0}) = \mathbb{R}$, $\Lambda^1 V = \pi(V) = V$.

   Let $v_1, \dots, v_n$ be a basis of V. Then $\Lambda^n V$ is clearly spanned by $v_1 \wedge \cdots \wedge v_n$, so $\dim \Lambda^n V \leq 1$. As the determinant is a nontrivial element of $\operatorname{Alt}^n(V)$, we have $\dim \Lambda^n V \geq 1$.

3. Clearly B spans $\Lambda^r V$. To show linear independence, suppose $\sum_I a_I v_I = 0$ (where I runs over all $(i_1 < \cdots < i_r)$ and $v_I = v_{i_1} \wedge \cdots \wedge v_{i_r}$). Fix I' and consider $J = I'^c$. Then

   $$0 = \left(\sum_I a_I v_I \right) \wedge v_J = \sum_I a_I v_I \wedge v_J = \pm a_{I'}.$$

   So $a_{I'} = 0$, proving linear independence.

□

Remark 9.16. Given a linear map $f\colon V \to W$, we get a map $f\colon T(V) \to T(W)$, which descends to give a map $\Lambda^r V \to \Lambda^r W$, called $\Lambda^r f$ or just f. It is given by

$$\Lambda^r f(v_1 \wedge \cdots \wedge v_r) = f(v_1) \wedge \cdots \wedge f(v_r)$$

and linear extension.

Lemma 9.17. *If* $\dim V = \dim W = n$, *then* $\Lambda^n f$ *is multiplication by* $\det f$, *i.e. if* $v_1, \dots, v_n$ *is a basis for* V *and* $w_1, \dots, w_n$ *one for* W, *then*

$$f(v_1 \wedge \cdots \wedge v_n) = \det(f) w_1 \wedge \cdots \wedge w_n.$$

Proof. Say $f(v_i) = \sum_j A_{ji} w_j$. Then

$$\begin{aligned} f(v_1) \wedge \cdots \wedge f(v_n) &= \left(\sum A_{j1} w_j \right) \wedge \cdots \wedge \left(\sum A_{jn} w_j \right) \\ &= \sum_{\pi \in S_n} A_{\pi(1)1} w_{\pi(1)} \wedge \cdots A_{\pi(n)n} w_{\pi(n)} \\ &= \underbrace{\sum_{\pi \in S_n} \varepsilon(\pi) A_{\pi(1)1} \cdots A_{\pi(n)n}}_{=\det(f)} w_1 \wedge \cdots \wedge w_n. \end{aligned}$$

□

Remark 9.18. Given a vector bundle $E \to M$, we have vector bundles $\Lambda^r E$ such that $(\Lambda^r E)_p = \Lambda^r E_p$ for all $p \in M$.

Definition 9.19. The *bundle of p-forms on M* is $\Lambda^p T^* M$. A section $\omega \in C^\infty(M, \Lambda^p T^* M)$ is called a *p-form on M*. $\Omega^p(U) = C^\infty(U, \Lambda^p T^* M)$ is the space of p-forms on U.

Remark 9.20. $\Omega^0(U) = C^\infty(U)$, $\Omega^1(U) = C^\infty(U, T^* M)$.

If we have coordinates $x^1, \ldots, x^n$ on U, then $\omega \in \Omega^p(U)$ can be written uniquely as

$$\omega = \sum_{i_1 < \cdots < i_r} a_{i_1, \ldots, i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

for $a_{i_1, \ldots, i_r}$ in $C^\infty(U)$.

Definition 9.21. Let $f\colon M \to N$ be a smooth map. The *pullback map* $f^*\colon \Omega^p(N) \to \Omega^p(M)$ is given by

$$f^*\omega|_x = \Lambda^p(Df|_x)^*(\omega|_{f(x)})$$

for $x \in M$, $\omega \in \Omega^p(N)$.

Example 9.22. $f^*\omega$ is smooth.

Remark 9.23. For $x \in M$:

$$\begin{gathered} Df|_x : T_x M \to T_{f(x)} N \\ (Df|_x)^* \colon T^*_{f(x)} N \to T^*_x M \\ \Lambda^p(Df|_x)^* \colon \Lambda^p T^*_{f(x)} N \to \Lambda^p T^*_x M \end{gathered}$$

Proposition 9.24.

1. *$f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$;*
2. *$f^*(\omega_1 \wedge \omega_2) = f^*(\omega_1) \wedge f^*(\omega_2)$;*
3. *$(g \circ f)^* = f^* \circ g^*$.*

Proof. not given □

10 Differential forms

Remark 10.1. If $\omega \in \Omega^q(U)$ and $x^1, \ldots, x^n$ are coordinates on U, then

$$\omega = \sum_{i_1 < \cdots < i_q} a_{i_1, \ldots, i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}, \qquad a_{i_1, \ldots, i_q} \in C^\infty(U).$$

If $y^1, \ldots, y^n$ are another set of coordinates on U, then $dx^{i_k} = \sum_{j=1}^n \frac{\partial x^{i_k}}{\partial y^j} dy^j$, so

$$\omega = \sum_{i_1 < \cdots < i_q} \sum_{1 \le j_1, \ldots, j_q \le n} a_{i_1, \ldots, i_q} \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_q}}{\partial y^{j_q}} dy^{j_1} \wedge \cdots \wedge dy^{j_q}.$$

Remark 10.2. If $f \in C^\infty(M) = \Omega^0(M)$, then $df \in \Omega^1(M)$. Locally: $df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n$.

Theorem 10.3. *There exist unique natural linear maps*

$$d\colon \Omega^q(M) \to \Omega^{q+1}(M)$$

($q \geq 0$), called the exterior derivative*, such that*

1. *if $f \in \Omega^0(M)$, then df is the derivative of f as defined previously;*

2. *$d^2 = 0$;*

3. *if $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, then $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.*

Proof. Say $\omega = \sum_{i_1 < \cdots < i_q} a_{i_1,\ldots,i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}$ locally. Set

$$d\omega = \sum_{i_1 < \cdots < i_q} da_{i_1,\ldots,i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} = \sum_{i_1 < \cdots < i_q} \sum_{j=1}^{n} \frac{\partial a_{i_1,\ldots,i_q}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}.$$

Then

$$d^2\omega = \sum_{i_1 < \cdots < i_q} \sum_{j=1}^{n} \frac{\partial a_{i_1,\ldots,i_q}}{\partial x^j \partial x^k} dx^j \wedge dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} = 0$$

as the partial derivatives commute. Further suppose $\alpha = f dx^I$, $\beta = g dx^J$ for some multiindices I, J with $|I| = p$, $|J| = q$. Then

$$\begin{aligned} d(\alpha \wedge \beta) &= d(fg dx^I \wedge dx^J) = d(fg) \wedge dx^I \wedge dx^J \\ &= g df \wedge dx^I \wedge dx^J + f d(g) \wedge dx^I \wedge dx^J = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta. \end{aligned}$$

So linearity gives 3.

For uniqueness and well-defined: Say $\omega = \sum_{i_1 < \cdots < i_q} b_{i_1,\ldots,i_q} dy^{i_1} \wedge \cdots \wedge dy^{i_q}$ for another set of coordinates. Set $\widetilde{d}\omega = \sum_{i_1 < \cdots < i_q} db_{i_1,\ldots,i_q} dy^{i_1} \wedge \cdots \wedge dy^{i_q}$. Then

$$d\omega \stackrel{3}{=} \sum_{i_1 < \cdots < i_q} db_{i_1,\ldots,i_q} dy^{i_1} \wedge \cdots \wedge dy^{i_q} + b_{i_1,\ldots,i_q} \underbrace{d(dy^{i_1} \wedge \cdots \wedge dy^{i_q})}_{=0 \text{ by 2 and 3}} = \widetilde{d}\omega.$$ □

Definition 10.4. Let $\omega \in \Omega^q(M)$ and let $X \in \mathrm{Vect}(M)$ with flow φ_t. The *Lie derivative* of ω is

$$L_X \omega = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* \omega.$$

Here $\varphi_t \colon M \to M$ and hence $\varphi_t^* \colon \Omega^q(M) \to \Omega^q(M)$.

Proposition 10.5.

1. *$L_X(\omega_1 + \omega_2) = L_X \omega_1 + L_X \omega_2$;*

2. *$L_X(f\omega) = X(f)\omega + f L_X(\omega)$;*

3. *$L_X(\omega(Y)) = (L_X \omega)(Y) + \omega(L_X Y)$ for $\omega \in \Omega^1(M)$ and $Y \in \mathrm{Vect}(M)$.*

Proof. exercise. □

11 De Rham cohomology

Definition 11.1. The *q-th de Rham cohomology group* of M is

$$H^q_{dR}(M) = \frac{\ker(d \colon \Omega^q \to \Omega^{q+1})}{\mathrm{im}(d \colon \Omega^{q-1} \to \Omega^q)}, \qquad q \geq 0.$$

Remarks 11.2.

1. $H^q_{dR}(M)$ is an $\mathbb{R}$-vector space.

2. $H^0_{dR}(M) = \ker(d\colon C^\infty(M) \to \Omega^1(M))$ is the space of locally constant functions. If M is connected, then $H^0_{dR}(M) = \mathbb{R}$.

3. If $d\alpha = 0$, the α is called *closed*.

4. If $\alpha = d\beta$, the α is called *exact*.

5. If $[\alpha] \in H^p_{dR}(M)$ and $[\beta] \in H^q_{dR}(M)$, define $[\alpha]\cdot[\beta] = [\alpha \wedge \beta] \in H^{p+q}_{dR}(M)$. (This is well-defined, since $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p \alpha \wedge d\beta = 0$ and $(\alpha + d\alpha')\wedge\beta = \alpha\wedge\beta + d(\alpha'\wedge\beta)$ by adding $(-1)^{p-1}\alpha' \wedge d\beta = 0$.)

6. *Functoriality:* Suppose $F\colon M \to N$ is smooth. Then $d \circ F^* = F^* \circ d$, so if $d\alpha = 0$, then $dF^*\alpha = 0$. Hence we can define

$$F^*\colon H^q_{dR}(N) \to H^q_{dR}(M), \qquad [\alpha] \mapsto [F^*\alpha].$$

Theorem 11.3 (de Rham)**.** *There exists a natural isomorphism* $H^q_{dR}(M) \cong H^q_{sing}(M)$.

Proof. sketch later. □

Definition 11.4. Let F and G be smooth maps $M \to N$. Then F and G are *smoothly homotopic* if there is a smooth map $H\colon M\times[0,1] \to N$ such that $H_0 = F$ and $H_1 = G$ (where we write $H_t(x) = H(x,t)$).

Theorem 11.5. *If F and G are smoothly homotopic, then $F^* = G^*\colon H^q_{dR}(N) \to H^q_{dR}(M)$ for all $q \geq 0$.*

Proof. Let $\omega \in \Omega^p(N)$ with $d\omega = 0$. Consider $H^*\omega$ on $M \times [0,1]$. We can write

$$H^*\omega = \sigma + dt \wedge \gamma,$$

where t is the coordinate on $[0,1]$, $\sigma = \sigma(t) \in \Omega^p(M)$ and $\gamma \in \Omega^{p-1}(M)$.

We will first show that $\sigma(t) = H_t^*\omega$: Fix t_0 and consider the embedding $\iota\colon M \times \{t_0\} \hookrightarrow M \times [0,1]$. Then

$$H^*_{t_0}\omega = (H\circ\iota)^*\omega = \iota^*H^*\omega = \iota^*(\sigma + dt\wedge\gamma) = \iota^*\sigma + \underbrace{\iota^* dt}_{=d\iota^*t = dt_0 = 0} \wedge \iota^*\gamma.$$

As $d\omega = 0$,

$$0 = H^*d\omega = dH^*\omega = d_{M\times[0,1]}(\sigma + dt\wedge\gamma) = d_M\sigma + \frac{\partial\sigma}{\partial t}dt - dt\wedge d\gamma.$$

Hence $\frac{\partial\sigma}{\partial t} = d\gamma$ and

$$G^*\omega - F^*\omega = H_1^*\omega - H_0^*\omega = \sigma(1) - \sigma(0) = \int_0^1 \frac{\partial\sigma}{\partial t}dt = \int_0^1 d\gamma dt = d\int_0^1 \gamma dt =: d\alpha.$$

Therefore $[F^*\omega] = [G^*\omega]$, i.e. $F^* = G^*$. □

Example 11.6. Consider $M = N = \mathbb{R}^n$ and $H_t(x) = tx$, i.e. $H_1 = \mathrm{id}$ and $H_0 \equiv 0$. Then $H_0^* = H_1^*\colon H^p_{dR}(\mathbb{R}^n) \to H^p_{dR}(\mathbb{R}^n)$. But for $p > 0$, $H_0^* = 0$ and $H_1^* = \mathrm{id}$. Hence $H^p_{dR}(\mathbb{R}^n) = 0$ for $p > 0$.

Corollary 11.7 (Poincaré Lemma)**.** *If M is a star-shaped domain, then $H^p_{dR}(M) = 0$ for all $p > 0$, i.e. if $\alpha \in \Omega^p(M)$ with $d\alpha = 0$, then there is a $\beta \in \Omega^{p-1}(M)$ with $\alpha = d\beta$.*

12 Integration

Definition 12.1. An *orientation* on a manifold M is an equivalence class of charts (U_α, x_α) such that $J(x_\alpha x_\beta^{-1}) > 0$ for all α, β. A manifold M is *orientable* if an orientation exists. An *oriented manifold* is a manifold with a chosen orientation.

Example 12.2. $\mathbb{R}^n$ and S^n are orientable. Note that

$$\mathbb{R}^n \to \mathbb{R}^n \colon (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$$

$$\mathbb{R}^n \to \mathbb{R}^n \colon (x_1, \ldots, x_n) \mapsto (x_1, \ldots, -x_n)$$

give two orientations on $\mathbb{R}^n$.

The Möbius band is not orientable.

Definition 12.3. Let ω be a section of $\Lambda^n T^* M$ (not necessarily smooth) and M oriented. If $\operatorname{supp}(\omega)$ is compact and contained in some chart U, $\omega = f dx^1 \wedge \ldots \wedge dx^n$ (f not necessarily smooth), define

$$\int_U \omega = \int_{x(U)} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n,$$

if the latter is defined.

(This is well-defined as $\operatorname{supp} f(x_1, \ldots, x_n)$ is compact and so the change of variable formula for $\int$ can be used and works because of the fixed orientation.)

Definition 12.4. Let $\{U_\alpha\}$ be an open cover of M. A collection of $\varphi_i \in C^\infty(M)$ is called a *partition of unity* (p.o.u.) subordinate to $\{U_\alpha\}$ if

1. For all i there exists $\alpha(i)$ such that $\operatorname{supp} \varphi_i \subseteq U_{\alpha(i)}$;
2. the $\operatorname{supp} \varphi_i$ are locally finite, i.e. for all $p \in M$, $\varphi_j(p) = 0$ for all but finitely many j; and
3. $\sum_i \varphi_i \equiv 1$ and each $\varphi_i \geq 0$.

Theorem 12.5. *Partitions of unity exist.*

Proof. Handout (only examinable if M is compact). □

Definition 12.6. Let ω be an n-form on M ($\dim M = n$, ω not necessarily smooth). Cover M by charts (U_α) and pick a subordinate partition of unity such that each φ_i has compact support. Define:

$$\int_M \omega = \sum_i \int_{U_{\alpha(i)}} \varphi_i \omega.$$

Theorem 12.7 (Stokes)**.** *If M is compact and $\alpha \in \Omega^{n-1}(M)$, then*

$$\int_M d\alpha = 0.$$

Proof. See the more general version below. □

Proposition 12.8. *A manifold M is orientable of and only if there exists $\omega \in \Omega^n(M)$ with $\omega_p \neq 0$ for all $p \in M$.*

Proof. First suppose that M is orientable. Given any chart (U, x) consider $\omega_U = dx^1 \wedge \cdots \wedge dx^n$ on U. If $y^1, \ldots, y^n$ is another chart of the same orientation, then $\omega_U = f dy^1 \wedge \cdots \wedge dy^n$ with f positive. Cover M by oriented charts (U_β, x_β) and choose a partition of unity φ_i subordinate to this cover. Set

$$\omega = \sum_i \varphi_i \omega_{U_{\alpha(i)}},$$

which is positive everywhere.

Conversely let $\omega \in \Omega^n(M)$ be nowhere vanishing. For a connected chart (U, x) we have $\omega = f dx^1 \wedge \cdots \wedge dx^n$ on U with f nowhere vanishing. Thus $f > 0$ or $f < 0$. In the first case, take $(U, (x^1, \ldots, x^n))$ as an oriented charts, otherwise take $(U, (x^1, \ldots, x^{n-1}, -x^n))$. □

Corollary 12.9. *If M is compact and orientable, then $H^n_{dR}(M) \neq 0$.*

Proof. Let $\omega \in \Omega^n(M)$ be nowhere vanishing. Without loss of generality $\omega > 0$. Clearly $d\omega = 0$. If $\omega = d\tau$, then

$$0 < \int_M \omega = \int d\tau = 0$$

by Stokes' theorem. This is absurd. Hence $[\omega] \neq 0$ in $H^n_{dR}(M)$. □

Definition 12.10. Let $\mathbb{H}^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. A *chart-with-boundary* on M is a bijection $x\colon U \to V \subseteq \mathbb{H}^+$, $U \subseteq M$, $V \subseteq \mathbb{H}^+$ open. Two charts are compatible if $x \circ y^{-1}$ is smooth. A *differentiable structure* is an equivalence class as before. This gives a *manifold-with-boundary.* Set

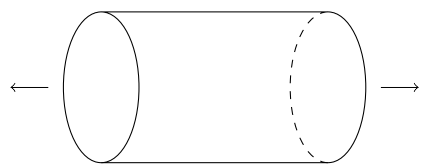
$$\partial M = \{p \in M : x_n(p) = 0 \text{ for some chart } (x^1, \ldots, x^n) \text{ around } p\}.$$

Example 12.11. $M = \overline{B_1(0)} \subseteq \mathbb{R}^n$, $\partial M = S^{n-1}$.

Fact 12.12. *If M is a manifold-with-boundary of dimension n, then ∂M is a manifold of dimension $n-1$ and $M^\circ = M \setminus \partial M$ is a manifold of dimension n.*

Definition 12.13. Suppose M is oriented, with the orientation given by some nowhere vanishing $\omega \in \Omega^n(M)$. Then we get the *induced orientation on ∂M* as follows: Say $x^1, \ldots, x^n$ is a chart such that $\partial M = \{x^n = 0\}$. If $\omega = f dx^1 \wedge \cdots \wedge dx^n$, we set $-f dx^1 \wedge \cdots \wedge dx^{n-1}$ to be the $(n-1)$-form on ∂M defining the induced orientation.

Example 12.14. Open cylinder: $\partial M = S^1 \amalg S^1$:



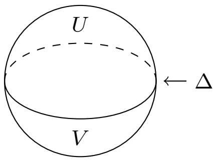
Theorem 12.15 (Stokes'). *If $\tau \in \Omega^{n-1}(M)$, then*

$$\int_M d\tau = \int_{\partial M} \tau.$$

Sketch of proof. By linearity we may assume that $\operatorname{supp}(\tau)$ is in some single chart and has compact support. So this reduces to Stokes' theorem in $\mathbb{R}^n$. □

Facts 12.16.

1. $\int_M \omega = \int_{M^\circ} \omega$.



2. *Suppose* $M = U \cup V \cup \Delta$, *where* U *and* V *are open submanifolds with* $U \cap V = \emptyset$ *and* Δ *has measure* 0 *in* M. *Then*
$$\int_M \omega = \int_U \omega + \int_V \omega.$$

3. *If* $M \subseteq \mathbb{R}^n$ *is a closed subset that is a manifold-with-boundary,* $\omega \in \Omega^n(R^n)$, $\omega = f dx_1 \wedge \cdots \wedge dx_n$, *then*
$$\int_M \omega = \int_M f\, dx_1 \cdots dx_n,$$
where the second integral denotes the usual integral for subsets of $\mathbb{R}^n$.

13 Connections

13.1 Connections as derivations

Let E be a vector bundle on M. Set $\Omega^p(E) = C^\infty(M, \Lambda^p T^*M \otimes E)$, the *$E$-valued p-forms.*

Definition 13.1. A *connection* on E is a linear map $\nabla \colon \Omega^0(E) \to \Omega^1(E)$ such that for all $f \in C^\infty(M)$ and all $s \in \Omega^0(E) = C^\infty(E)$,
$$\nabla(fs) = df \otimes s + f\nabla s.$$

This gives a way to differentiate sections of E along vector fields: For $X \in \mathrm{Vect}(M)$ set
$$\nabla_X(s) = (\nabla s, X) := C(\nabla s \otimes X),$$
where the contraction is between the 1-form part of ∇s and the vector field. So $\nabla_X(fs) = (df \otimes s, X) + (f\nabla s, X)$ and hence we have
$$\begin{aligned}\nabla_X(fs) &= X(f)s + f\nabla_X s\\ \nabla_{gX}(s) &= g\nabla_X(s)\\ \nabla_{X_1+X_2}(s) &= \nabla_{X_1}s + \nabla_{X_2}s.\end{aligned}$$

Example 13.2. Consider $U \subseteq_o \mathbb{R}^n$ and $E = U \times \mathbb{R}^m$. Define sections $s_1, \dots, s_m$ by $s_i(x) = (x, (0, \dots, 1, \dots, 0))$ and define ∇ by requiring $\nabla s_i = 0$ for all i. For any section $s \in C^\infty(E)$ write $s = \sum a_i s_i$ uniquely for some $a_i \in C^\infty(M)$. Then
$$\nabla s = \sum da_i \otimes s_i + \underbrace{a_i \nabla s_i}_{=0}.$$

If $X \in \mathrm{Vect}(U)$, then $\nabla_X(s) = \sum X(a_i)s_i$. We write $s = (a_1, \dots, a_n)$ and $\nabla_X(s) = (X(a_1), \dots, X(a_m))$.

Exercise 13.3. Show that every vector bundle admits at least one connection.

Lemma 13.4. *If* ∇_1 *and* ∇_2 *are connections on* E, *then* $\nabla_1 - \nabla_2 \in \Omega^1(M, E \otimes E^*) = \Omega^1(\mathrm{Hom}(E, E))$.

Proof. For any $s \in C^\infty(E)$ and $g \in C^\infty(M)$:
$$(\nabla_1 - \nabla_2)(gs) = \nabla_1(gs) - \nabla_2(gs) = dg \otimes s + g\nabla_1 s - dg \otimes s - g\nabla_2 s = g(\nabla_1 - \nabla_2)s.$$

So $\nabla_1 - \nabla_2 \colon C^\infty(E) \to C^\infty(T^*M \otimes E)$ is $C^\infty(M)$-linear. Hence it is induced by a bundle map $E \to T^*M \otimes E$, i.e. given by an element of $\Omega^1(M, E \otimes E^*)$. ☐

Remark 13.5. Given a connection ∇ on E, we get an induced map

$$d_\nabla \colon \Omega^p(E) \to \Omega^{p+1}(E), \qquad d_\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s$$

for $\omega \in \Omega^p(M)$ and $s \in C^\infty(E)$, called the *covariant derivative.* One checks that for $\zeta \in \Omega^q(E)$,

$$d_\nabla(\omega \wedge \zeta) = d\omega \wedge \zeta + (-1)^p \omega \wedge d_\nabla \zeta.$$

13.2 Coordinate description of connections

Definition 13.6. Sections $s_1, \ldots, s_m \in \Omega^0(U, E)$ are a *frame* for E if for all $p \in U$, $s_1(p), \ldots, s_m(p)$ is a basis of E_p.

Suppose we have a frames $e_1, \ldots, e_m$ for E and $\omega^1, \ldots, \omega^l$ for $\Lambda^p T^* M$ on U. Given $s \in \Omega^p(E)$, we can write $s = \sum_{i,j} f^i_j \omega^j \otimes e_i$ with $f^i_j \in C^\infty(U)$.

Exercise 13.7. The quantities $\sigma^i = \sum_j f^i_j \omega^j$ are independent of the chosen frame.

Hence we can write $s = \sum_i \sigma^i \otimes e_i$ with $\sigma^i \in \Omega^p(U)$, $e_i \in \Omega^0(U, E)$ on U.

Definition 13.8. Given a connection ∇, write $\vartheta e_j = \sum_i \vartheta^i_j e_i$ for some $\vartheta^i_j \in \Omega^1(U)$. The matrix $\vartheta = (\vartheta^i_j)_{i,j}$ of 1-forms is called the *connection matrix* of ∇ with respect to the frame $e_1, \ldots, e_m$.

Lemma 13.9. *Let $s \in \Omega^0(U, E)$, $s = \sum_i s^i e_i$. Write $\mathsf{s} = (s^1, \ldots, s^m)^T$, $\mathsf{ds} = (ds^1, \ldots, ds^m)^T$ and $\mathsf{e} = (e_1, \ldots, e_m)^T$. Then*

$$\nabla s = (\mathsf{ds} + \vartheta \mathsf{s}) \cdot \mathsf{e}.$$

Proof.

$$\begin{aligned}
\nabla s &= \sum_i \nabla(s^i e_i) && \text{(linearity)} \\
&= \sum_i \left(ds^i \otimes e_i + s^i \nabla e_i\right) && \text{(Leibnitz)} \\
&= \sum_i \left(ds^i \otimes e_i + s^i \sum_j \vartheta^j_i e_j\right) \\
&= \sum_j \left(ds^j + \sum_i s^i \vartheta^j_i\right) \otimes e_j
\end{aligned}$$

□

Proposition 13.10. *Suppose that $e_1, \ldots, e_m$ and $e'_1, \ldots, e'_m$ are frames of E over U. Let ∇ be a connection that has corresponding connection matrices ϑ and ϑ'. Let $\psi \colon U \to \mathrm{GL}_m(\mathbb{R})$ such that $\mathsf{e}' = \psi \mathsf{e}$. Then*

$$\vartheta' = (d\psi)\psi^{-1} + \psi \vartheta \psi^{-1}.$$

Proof.

$$\begin{aligned}
\nabla \mathsf{e}' = \nabla(\psi \mathsf{e}) &= (d\psi)\mathsf{e} + \psi \nabla \mathsf{e} \\
&= (d\psi)\mathsf{e} + \psi(\vartheta \mathsf{e}) \\
&= (d\psi)\psi^{-1}\mathsf{e}' + \psi \vartheta \psi^{-1} \mathsf{e}'.
\end{aligned}$$

□

Remarks 13.11.

1. If $\nabla = d + \vartheta$, then $d_\nabla \colon \Omega^p(U,E) \to \Omega^{p+1}(U,E)$ can also be written as $d_\nabla = d + \vartheta$, i.e. $d_\nabla \mathsf{s} = d\mathsf{s} + \vartheta \wedge \mathsf{s}$ for $s \in \Omega^p(U,E)$.

2. For a given frame $e_1, \dots, e_m$ for E over U and coordinates $x^1, \dots, x^n$ on U, we can write $\vartheta^i_j = \sum_k \Gamma^i_{jk} dx^k$ for some $\Gamma^i_{jk} \in C^\infty(U)$ $(1 \le k \le n,\ 1 \le i,j \le m)$ called *Christoffel symbols* of ∇.

13.3 Connections and parallel transport

Definition 13.12. A smooth function $s\colon [0,1] \to E$ is a *section of E along a curve* $c\colon [0,1] \to M$, if $s(t) \in E_{c(t)}$ for all t. The space of all sections along c is denoted $C^\infty(c,E)$.

If $s \in \Omega^0(E)$, then $\widetilde{s}(t) = s(c(t))$ is a section along c. We will also write $s(t)$ for this section. Note that not all elements of $C^\infty(c,E)$ look like this.

Example 13.13. $\dot{c}(t) \in C^\infty(c,TM)$, but if c has a node, then this cannot be the restriction of any $s \in \Omega^0(TM)$.

Proposition 13.14. *There exists a unique* covariant derivative map

$$\frac{D}{dt}\colon C^\infty(c,E) \to C^\infty(c,E),$$

depending on c, such that

1. *$\frac{D}{dt}$ is linear;*

2. *if $f\colon [0,1] \to \mathbb{R}$ is smooth and $s \in C^\infty(c,E)$, then $\frac{D}{dt}(fs) = \frac{df}{dt}s + f\frac{Ds}{dt}$.*

3. *if $s \in \Omega^0(E)$, then $\frac{D}{dt}(s(t)) = \nabla_{\dot{c}(t)} s$.*

Proof. Work locally, say with a frame $e_1, \dots, e_m$ of E and coordinates $x^1, \dots, x^n$ on U. If $s \in C^\infty(c,E)$, we can write $s = \sum s^i(t)\, e_i|_{c(t)}$ for some $s^i \in C^\infty([0,1])$. Assuming that $\frac{D}{dt}$ exists, we calculate

$$\begin{aligned} \frac{D}{dt}s &\overset{1.}{=} \sum_i \frac{D}{dt}\left(s^i(t)\, e_i|_{c(t)}\right) \\ &\overset{2.}{=} \sum_i \left(\frac{ds^i}{dt}\, e_i|_{c(t)} + s^i \frac{D}{dt}(e_i \circ c)\right) \\ &\overset{3.}{=} \sum_i \left(\frac{ds^i}{dt}\, e_i|_{c(t)} + s^i \nabla_{\dot{c}(t)}\, e_i|_{c(t)}\right). \end{aligned}$$

Let $c = (c^1, \dots, c^n)$. Then $\dot{c} = \sum_j \dot{c}^j\, \frac{\partial}{\partial x^j}\big|_{c(t)}$ So,

$$\nabla_{\dot{c}(t)} e_i\big|_{c(t)} = \sum_j \dot{c}^j\, \nabla_{\frac{\partial}{\partial x^j}} e_i\Big|_{c(t)} = \sum_{j,k} \dot{c}^j\, \Gamma^k_{ij}\Big|_{c(t)}\, e_k|_{c(t)}\,.$$

Hence

$$\frac{D}{dt}(s) = \sum_k \left(\frac{ds^k}{dt} + \sum_{i,j} s^i \dot{c}^j (\Gamma^k_{ij} \circ c)\right) e_k|_{c(t)} \tag{4}$$

determines $\frac{D}{dt}$ uniquely. $\square$

Definition 13.15. We say $s \in C^\infty(c, E)$ is *parallel*, if $\frac{Ds}{dt} = 0$ for all $t \in [0,1]$.

By (4), this condition is a system of linear ODEs for s^i. So if $v \in E_{c(0)}$ is an initial condition, the theory of ODEs says that there is a unique solution $s \in C^\infty(c, E)$ with $\frac{Ds}{dt} \equiv 0$ and $s(0) = v$. This s is called the *parallel transport of v along c*.

If we set $a = c(0)$ and $b = c(1)$, we have linear isomorphisms

$$P^c_{a,b} \colon E_a \to E_b, \qquad P^c_{a,b}(v) = s(1),$$

where $s \in C^\infty(c, E)$ is a parallel transport of v.

Given $P^c_{a,b}$, we can recover ∇:

Proposition 13.16. *Let $c \colon (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$, $\dot{c}(0) = v \in T_pM$ and write $P_t = P^c_{p,c(t)} \colon E_p \xrightarrow{\sim} E_{c(t)}$. Then, for $s \in \Omega^0(E)$,*

$$\nabla_v s = \lim_{t\to 0} \frac{P_t^{-1}(s|_{c(t)}) - s|_p}{t}.$$

Proof. Let $e_1, \dots, e_m$ be a basis for E_p. Define $\widetilde{e}_i(t) = P_t(e_i)$ Since P_t is an isomorphism for all t, $\widetilde{e}_1, \dots, \widetilde{e}_m$ are a frame along c (called a *parallel frame*). Given s, we write $s = \sum s^i(t)\widetilde{e}_i(t)$ and calculate

$$\begin{aligned}\frac{P_t^{-1}(s|_{c(t)}) - s|_p}{t} &= \frac{\sum_i \left(s^i(t)P_t^{-1}(\widetilde{e}_i(t)) - s^i(0)\widetilde{e}_i(0)\right)}{t} \\ &= \sum_i \frac{s^i(t) - s^i(0)}{t} e_i \xrightarrow{t\to 0} \sum_i \dot{s}^i(0) e_i = \nabla_{\dot{c}(0)} s = \nabla_v s. \quad \square\end{aligned}$$

13.4 Horizontal splitting

See notes.

14 Curvature

Definition 14.1. The *curvature* of a connection ∇ is the map

$$\mathcal{R} = d_\nabla \circ d_\nabla \colon \Omega^0(E) \to \Omega^2(E).$$

Lemma 14.2. *$\mathcal{R}$ is $C^\infty(M)$-linear.*

Proof. Let $f \in C^\infty(M)$, $s \in \Omega^0(E)$.

$$\mathcal{R}(fs) = d_\nabla d_\nabla(fs) = d_\nabla\left(df \otimes s + f d_\nabla s\right) = \underbrace{d^2 f}_{=0} \otimes s - df \otimes d_\nabla s + df \otimes d_\nabla s + f d_\nabla d_\nabla s = f\mathcal{R}(s).$$

$\square$

Remark 14.3. So $\mathcal{R}$ induces a bundle morphism $E \to E \otimes \Lambda^2 T^*M$, and thus a section

$$R \in \Omega^0(E^* \otimes E \otimes \Lambda^2 T^*M) = \Omega^2(E \otimes E^*) = \Omega^2(\mathrm{End}(E)).$$

We can write this as $\mathcal{R}(s) = R \wedge s \in \Omega^2(E)$, where we contract the endomorphism part with s.

Remark 14.4. In coordinates: Let $e_1, \dots, e_m$ be a frame for E over U and let ∇ have connection matrix $\vartheta = (\vartheta^i_j)_{i,j}$ ($\vartheta^i_j \in \Omega^1(U)$) defined by $\nabla e_j = \sum_i \vartheta^i_j e_i$. Then the *curvature matrix* Θ of ∇ with respect to this frame is defined by

$$\mathcal{R}(e_j) = \sum_i \Theta^i_j \otimes e_i, \qquad \Theta^i_j \in \Omega^2(M).$$

Lemma 14.5. $\Theta = d\vartheta + \vartheta \wedge \vartheta$.

Proof.

$$\begin{aligned}\mathcal{R}(e_j) = d_\nabla d_\nabla (e_j) &= d_\nabla \left(\sum_i \vartheta^i_j \otimes e_i \right) \\ &= \sum_i \left(d\vartheta^i_j \otimes e_j - \vartheta^i_j \wedge d_\nabla(e_i) \right) \\ &= \sum_i \left(d\vartheta^i_j \otimes e_j - \vartheta^i_j \wedge \sum_k \vartheta^k_i \otimes e_k \right) \\ &= \sum_k \left(d\vartheta^k_j - \sum_i \vartheta^i_j \wedge \vartheta^k_i \right) \otimes e_k. \end{aligned}$$

□

Proposition 14.6 (Bianchi identity)**.** $d\Theta = \Theta \wedge \vartheta - \vartheta \wedge \Theta$.

Proof.

$$\begin{aligned} d\Theta = d(d\vartheta + \vartheta \wedge \vartheta) &= \underbrace{d^2\vartheta}_{=0} + d\vartheta \wedge \vartheta - \vartheta \wedge d\vartheta \\ &= (\Theta - \vartheta \wedge \vartheta) \wedge \vartheta - \vartheta \wedge (\Theta - \vartheta \wedge \vartheta) = \Theta \wedge \vartheta - \vartheta \wedge \Theta. \end{aligned}$$

□

Definition 14.7. For $X, Y \in \mathrm{Vect}(M)$, define

$$\mathcal{R}(X,Y) \colon \Omega^0(E) \to \Omega^0(E), \qquad s \mapsto C(\mathcal{R}(s) \otimes X \otimes Y),$$

where the 2-form part of $\mathcal{R}(s)$ is contracted with X and Y.

Lemma 14.8. $\mathcal{R}(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.

Proof. We will show that given $\alpha \in \Omega^1(E)$,

$$(d_\nabla \alpha)(X,Y) = \nabla_X(\alpha(Y)) - \nabla_Y(\alpha(X)) - \alpha([X,Y]).$$

Then with $\alpha = d_\nabla(s)$ (thus $\alpha(Y) = \nabla_Y(s)$):

$$\mathcal{R}(X,Y)(s) = d_\nabla d_\nabla(s)(X,Y) = d_\nabla(\alpha)(X,Y) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

For the claim, we may assume by linearity that $\alpha = \omega \otimes s$ with $\omega \in \Omega^1(M)$, $s \in \Omega^0(E)$. Then

$$d_\nabla(\alpha)(X,Y) = d_\nabla(\omega \otimes s)(X,Y) = (d\omega \otimes s - \omega \wedge \nabla s)(X,Y) = d\omega(X,Y) \otimes s - \omega(X) \nabla_Y s + \omega(Y) \nabla_X s.$$

On example sheet 2: $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$. Thus

$$d_\nabla(\alpha)(X,Y) = X(\omega(Y))s - Y(\omega(X))s - \omega([X,Y])s - \omega(X)\nabla_Y s + \omega(Y) \nabla_X s.$$

Also

$$\begin{aligned}\nabla_X(\alpha(Y)) &= \nabla_X((\omega\otimes s)(Y))\\ &= \nabla_X(\omega(Y)s) + (\nabla(\omega(Y)s), X)\\ &= (d(\omega(Y))\otimes s + \omega(Y)\nabla s, X)\\ &= X(\omega(Y))s + \omega(Y)\nabla_X s\\ \nabla_Y(\alpha(X)) &= Y(\omega(X))s + \omega(X)\nabla_Y s\\ \alpha([X,Y]) &= \omega([X,Y])s. \qquad \square\end{aligned}$$

Example 14.9. Let $M = U$, $E = U\times\mathbb{R}^m$ with frame $e_1,\dots,e_m$. Define a connection on E by requiring $\nabla e_i = 0$ for all i. So if $s = \sum_j s^j e_j \in \Omega^0(E)$, then $\nabla s = \sum_j ds^j \otimes e_j$. Further, if $X \in \mathrm{Vect}(M)$, then $\nabla_X(s) = \sum_j X(s^j)e_j$. So clearly $(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})(s) = 0$.

Hence we can interpret $\mathcal{R}$ as a measure to which extent a vector bundle E is trivial.

Remark 14.10. Let $e_1,\dots,e_m$ and $e'_1,\dots,e'_m$ be frames on U with connection and curvature matrixes ϑ, Θ and ϑ', Θ'. Let $\mathsf{e}' = \psi\mathsf{e}$ for some $\psi\colon U \to \mathrm{GL}_m$. Then $\vartheta' = (d\psi)\psi^{-1} + \psi\vartheta\psi^{-1}$ and

$$\mathcal{R}'(\mathsf{e}') = \mathcal{R}(\psi\mathsf{e}) = \psi\mathcal{R}(\mathsf{e}) = \psi(\Theta\otimes\mathsf{e}) = \psi\Theta\psi^{-1}\mathsf{e}'.$$

Thus

$$\Theta' = \psi\Theta\psi^{-1}.$$

In particular, $\mathrm{tr}(\Theta) \in \Omega^2(E)$ is independent of the chosen frame.

Facts 14.11.

1. $d\,\mathrm{tr}(\Theta) = 0$. *Set* $c_1 := [\mathrm{tr}(\Theta)] \in H^2_{dR}(M)$.
2. c_1 *is independent of* ∇.

15 Linear connections

Definition 15.1. A connection on TM is called a *linear connection* or a *Koszul connection* or a *connection on* M.

Remark 15.2. Thus a linear connection is a map $\mathrm{Vect}(M)\times\mathrm{Vect}(M)\to\mathrm{Vect}(M)$ sending (X,Y) to $\nabla_X Y$. Remember that we have $\nabla_{fX}(Y) = f\nabla_X(Y)$ and $\nabla_X(fY) = X(f)Y + f\nabla_Y X$.

Definition 15.3. A *Riemannian metric* on M is a $g \in C^\infty(T^*M\otimes T^*M)$ such that

1. Symmetry: $g(X_p, Y_p) = g(Y_p, X_p)$ for all $X_p, Y_p \in T_pM$.
2. Non-degeneracy: If $g(X_p, Y_p) = 0$ for all X_p, then $Y_p = 0$.
3. Positive definite: $g(X_p, X_p) \geq 0$ for all X_p.

Write $\langle X, Y\rangle = g(X,Y)$. Given $X_p \in T_pM$, set $|X_p| = \langle X_p, X_p\rangle^{\frac{1}{2}}$. Given $c\colon [0,1]\to M$, set

$$\mathrm{length}(c) = \int_0^1 |\dot{c}(t)|dt.$$

Given $a, b \in M$, set

$$d(a,b) = \inf\{\mathrm{length}(c) : c\colon [0,1]\to M \text{ with } c(0) = a,\ c(1) = b\}.$$

Exercise 15.4. If M is connected, d gives a metric.

We want to show how g gives rise to a unique connection on M with nice properties.

Example 15.5. Let $M = \mathbb{R}^n$ with global coordinates $x^1, \ldots, x^n$. Then $TM = \mathbb{R}^n \times \mathbb{R}^n$ with global frame $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. Define a Riemannian metric by $g(X,Y) = \sum_i X^i Y^i$, where $X = \sum X^i \frac{\partial}{\partial x^i}$, $Y = \sum Y^i \frac{\partial}{\partial x^i}$. We also have a linear connection given by $\nabla \frac{\partial}{\partial x^i} = 0$ for all i, i.e. $\nabla_X Y = \sum X(Y^i) \frac{\partial}{\partial x^i}$. We observe

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \qquad \text{for all } X, Y, Z,$$

where we set $\nabla_X(f) = X(f)$.

Definition 15.6. A linear connection is *compatible* with a Riemannian metric g, if

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \qquad \text{for all } X, Y, Z \in \operatorname{Vect}(M).$$

Remark 15.7. Observe on $\mathbb{R}^n$: $\nabla_X Y - \nabla_Y X = [X, Y]$.

Definition 15.8. Given a linear connection ∇ on M, set the *torsion* of ∇ to be

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Say ∇ is *symmetric* (or *torsion-free*) if $\tau(X,Y) = 0$ for all $X, Y \in \operatorname{Vect}(M)$.

Theorem 15.9 (Fundamental theorem of Riemannian geometry)**.** *Let M be a manifold with Riemannian metric g. Then there exists a unique linear connection ∇ that is symmetric and compatible with g. It is called* Levi-Civita connection*.*

Proof. First show uniqueness: Suppose ∇ is symmetric and compatible with g:

$$\begin{aligned} X(\langle Y, Z \rangle) &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\ &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_Z X \rangle + \langle Y, [X, Z] \rangle & (5) \\ Y(\langle Z, X \rangle) &= \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_X Y \rangle + \langle Z, [Y, X] \rangle & (6) \\ Z(\langle X, Z \rangle) &= \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Y Z \rangle + \langle X, [Z, Y] \rangle & (7) \end{aligned}$$

(5) + (6) − (7):

$$X(\langle Y, Z \rangle) + Y(\langle Z, X \rangle) - Z(\langle X, Y \rangle) = 2 \langle \nabla_X Y, Z \rangle + \langle Y, [X, Z] \rangle + \langle Z, [Y, X] \rangle - \langle X, [Z, Y] \rangle.$$

Rearranging:

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left(X(\langle Y, Z \rangle) + Y(\langle Z, X \rangle) - Z(\langle X, Y \rangle) - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \right). \quad (8)$$

The right hand side is independent of ∇, hence it defines $\nabla_X Y$ uniquely in terms of the metric.

For existence use (8) to define $\nabla_X Y$ and check that it is a linear connection with the desired properties. (See handout.) □

Example 15.10. Suppose M is embedded into $\mathbb{R}^N$ via φ such that $D\varphi$ preserves the metric, i.e.

$$\langle D\varphi(X_p), D\varphi(Y_p) \rangle_{\mathbb{R}^N} = \langle X_p, Y_p \rangle_g \qquad \text{for all } p \in M \text{ and } X_p, Y_p \in T_p M.$$

Denote the standard connection on $\mathbb{R}^N$ by $\widetilde{\nabla}$ and set $\pi \colon T_p \mathbb{R}^N \to T_p M$ to be the orthogonal projection with respect to the metric on $\mathbb{R}^N$. Define ∇ on M by

$$\nabla_X Y \coloneqq \pi(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}), \qquad X, Y \in \operatorname{Vect}(M),\ \widetilde{X}, \widetilde{Y} \in \operatorname{Vect}(\mathbb{R}^N) \text{ extending } X, Y.$$

Exercise: This gives a well-defined connection that is symmetric and compatible with g.

Definition 15.11. Let $E \to M$, $E' \to M$ be two vector bundles with connections ∇ and ∇' respectively. Get *induced connections* $\widetilde{\nabla}$

1. on $E \otimes E'$ by $\widetilde{\nabla}(s \otimes s') = \nabla s \otimes s' + s \otimes \nabla' s'$;

2. on $E \oplus E'$ by $\widetilde{\nabla}(s, s') = (\nabla s, \nabla' s')$;

3. on E^*: for $\sigma \in C^\infty(E^*)$: $\nabla^*(\sigma)(s) = d(\sigma(s)) - \sigma(\nabla s)$, $s \in C^\infty(E)$.

Exercise 15.12. Check that these definitions indeed give connections. For the connection on the dual bundle:

$$\begin{aligned} \nabla^*(f\sigma)(s) &= d(f\sigma(s)) - f\sigma(\nabla s) \\ &= (df)\sigma(s) + fd(\sigma(s)) - f\sigma(\nabla s) \\ &= (df)\sigma(s) + f\nabla^*\sigma(s) \qquad\qquad \forall s \in C^\infty(E). \end{aligned}$$

Thus $\nabla^*(f\sigma) = df \otimes \sigma + f\nabla^*\sigma$.

So given a connection on TM, we get induced connections on $T^{(k,l)}M = (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$ with the convention that the induced connection on $T^{(0,0)}M = M \otimes \mathbb{R}$ is $\nabla f = df$, $f \in C^\infty(M)$.

Remark 15.13. For $X \in \mathrm{Vect}(M)$:

1. $\nabla_X(f) = X(f)$.

2. $\nabla_X(F \otimes G) = F \otimes \nabla_X G + \nabla_X F \otimes G$, for $F \in \mathcal{T}^{(k,l)}M$ and $G \in \mathcal{T}^{(k',l')}M$.

3. $d((\sigma, Y)) = (\nabla\sigma)(Y) + \sigma(\nabla Y)$ for $\sigma \in \Omega^q(M)$, $Y \in \mathrm{Vect}(M)$;
   $X(\sigma(Y)) = (\nabla_X\sigma)(Y) + \sigma(\nabla_X Y)$.

4. $\nabla_X(\omega, Y) = (\nabla_X\omega, Y) + (\omega, \nabla_X Y)$ for $\omega \in \Omega^p(M)$ and $Y \in \mathcal{T}^{(p,0)}M$.

Proposition 15.14. *Let $g \in C^\infty(T^*M \otimes T^*M)$ be a metric on M and ∇ a connection on M. The following are equivalent:*

1. *∇ is compatible with g.*

2. *$\nabla g = 0$.*

3. *Is V, W are vector fields along a curve c, then $\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{D}{dt}V, W\right\rangle + \left\langle V, \frac{D}{dt}W\right\rangle$.*

4. *Suppose V, W are parallel along a curve c, Then $\langle V, W\rangle$ is constant along c. Equivalently $|V|$ is constant along c.*

5. *Parallel transport is an isometry: Given $c\colon [0,1] \to M$, $c(0) = p$, $c(1) = q$, then $P^c_{p,q}\colon T_pM \to T_qM$ is an isometry of vector spaces.*

Proof. The identity

$$(\nabla_X g)(Y, Z) = \nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

implies the equivalence of 1 and 2. For 1 to 3 apply 1 to $\dot{c}(t)$:

$$\frac{d}{dt}\langle V, W\rangle = \nabla_{\dot{c}(t)}\langle V, W\rangle = \langle \nabla_{\dot{c}}V, W\rangle + \langle V, \nabla_{\dot{c}}W\rangle = \left\langle \frac{D}{dt}V, W\right\rangle + \left\langle V, \frac{D}{dt}W\right\rangle.$$

The implications 3 to 4 to 5 are obvious since parallel means $\frac{D}{dt}V = 0$. The implication 5 to 1 is an exercise. □

16 Geodesics and the exponential map

Definition 16.1. A *geodesic* on M is a curve $c\colon [0,1] \to M$ such that $\nabla_{\dot c(t)} \dot c(t) = 0$ for all $t \in [0,1]$. Equivalently $\dot c(t)$ is parallel along c; equivalently $\frac{D}{dt}\dot c(t) = 0$.

Proposition 16.2. *Geodesics exist locally and are unique given initial conditions $c(0)$, $\dot c(0)$.*

Proof. Recall the discussion of parallel transport, in particular formula (4). In this case $E = TM$, the frame is given by $\frac{\partial}{\partial x^i}$ and $s = \dot c$. So c is a geodesic exactly if

$$\sum_k \left(\ddot c^k(t) \sum_{i,j} \dot c^j(t) \dot c^i(t)\, \Gamma^k_{ij}\Big|_{c(t)} \right) \frac{\partial}{\partial x^k} = 0.$$

Hence we need to solve the system

$$\begin{cases} \alpha^k(t) = \dot c^k(t) \\ \dot\alpha^k(t) + \sum_{i,j} \alpha^j(t)\alpha^i(t)\, \Gamma^k_{ij}\Big|_{c(t)} = 0 \end{cases}$$

for $k = 1, \ldots, n$. This is a system of ODEs and hence locally has a unique solution. □

Definition 16.3. Fix a point $p \in M$. For $v \in T_pM$ let γ_v be the unique geodesic through p with $\dot\gamma_v(0) = v$. Define $\exp\colon S \subseteq TM \to M$ by $\exp(v) = \gamma_v(1)$ for $v \in T_pM$, where $S \subseteq TM$ is the set where $\gamma_v(1)$ is defined.

Fact 16.4. *From the theory of ODEs: S is open and* $\exp$ *is smooth. Clearly* $0 \in S$.

Lemma 16.5. $\exp(tv) = \gamma_v(t)$.

Proof. Fix t and set $c(s) = \gamma_v(st)$. By the chain rule, c is a geodesic with $\dot c(0) = t\dot\gamma_v(0) = tv$. So by uniqueness of geodesics, $\exp(tv) = c(1) = \gamma_v(t)$. □

Proposition 16.6. *Consider* $\exp$ *restricted to T_pM and let $S \subseteq T_pM$ be its domain. Then* $\exp$ *is a local diffeomorphism, i.e. there exist open neighborhoods U of 0 in S and U' of p in M such that* $\exp$ *is an isomorphism from U to U'.*

Proof. By the inverse function theorem is suffices to prove that

$$D\exp\colon T_0T_pM \cong T_pM \longrightarrow T_pM$$

is an isomorphism. We will even show that $D\exp = I$. Fix $v \in T_pM$ and consider the curve $\tau\colon t \to tv$ in T_pM. It has $\dot\tau(0) = v$ and

$$D\exp(v) = \frac{d}{dt}\Big|_{t=0} \exp(\tau(t)) = \frac{d}{dt}\Big|_{t=0} \exp(tv) = \frac{d}{dt}\Big|_{t=0} \gamma_v(t) = v. \qquad \square$$

Let $e_1, \ldots, e_n$ be an orthonormal basis of T_pM. Let the "coordinatization" with respect to this basis be α, i.e.

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow[\sim]{\alpha} & T_pM & & \\ \cup & & \cup & & \\ U_0 & \xrightarrow[\sim]{\alpha} & U & \xrightarrow[\sim]{\exp} & U' \subseteq M \\ & \xleftarrow{\hspace{5em}x\hspace{5em}} & & & \end{array}$$

Set $x = \alpha^{-1} \circ \exp^{-1}$ on U'. This is a coordinate chart on U such that

1. $x(p) = 0$;

2. geodesics through p are straight lines, i.e. the geodesic through p with initial vector $v = \sum v^i \frac{\partial}{\partial x^i}$ is mapped to the curve $\gamma_v(t) = (v^1 t, \ldots, v^n t)$;

3. If $g = \sum g_{ij} dx^i \otimes dx^j$, then $g_{ij}\big|_p = \delta_{ij}$;

Definition 16.7. such a chart x is called a *system of normal coordinates* (or *geodesic coordinates*) on M centered at p.

With $r = \sqrt{\sum (x^i)^2}$ define $\frac{\partial}{\partial r} = \sum \frac{x^i}{r} \frac{\partial}{\partial x^i}$ on $U' \setminus \{p\}$.

Proposition 16.8. *If $|v| = 1$, then $\gamma_v(t) = \exp(vt)$ is a unit speed geodesic with $\dot\gamma_v(t) = \frac{\partial}{\partial r}\big|_{\gamma(t)}$.*

Proof. As $\dot\gamma_v$ is parallel, $|\dot\gamma_v(t)|$ is independent of t and $|\dot\gamma_v(0)| = |v| = 1$. By 2 above, $\dot\gamma_v(t) = \sum v^i \frac{\partial}{\partial x^i}$, i.e. $\dot\gamma_v(x^j) = v^j$. Also $\frac{\partial}{\partial r}(x^j) = \frac{x^j}{r}$, so

$$\frac{\partial}{\partial r} x^j \bigg|_{\gamma(t)} = \frac{t v^j}{\left(\sum t^2 {v^i}^2\right)^{1/2}} = v^j$$

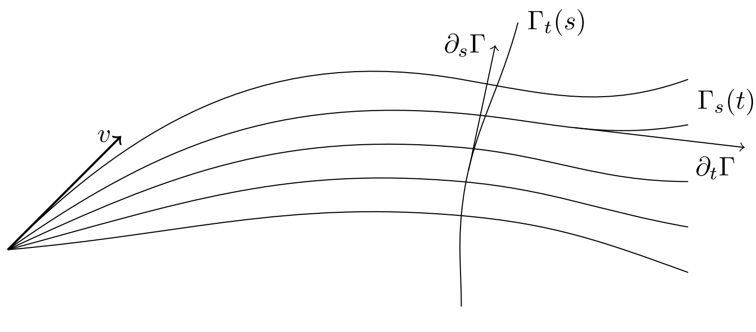
as $|v| = 1$. ∎

We want to study how geodesics change when the initial vector changes. For this we need study *variations*. Fix a point $p \in M$ and a vector $v \in T_pM$. Consider a curve σ in T_pM through v, i.e. $\sigma\colon (-\varepsilon, \varepsilon) \to T_pM$ with $\sigma(0) = v$. Set

$$\Gamma(s,t) = \exp(t\sigma(s)).$$

This is the geodesic starting at $\sigma(s)$ at time t. We have $\Gamma(0,t) = \exp(tv)$ and $\Gamma(s,0) = p$. Write $\Gamma_s(t) = \Gamma(s,t) = \Gamma_t(s)$ (the first is considered as a curve in t with fixed s and the second as a curve in s with fixed t). Define

$$\partial_t \Gamma = \frac{d}{dt}\Gamma_s \quad \text{and} \quad \partial_s \Gamma = \frac{d}{ds}\Gamma_t.$$



Lemma 16.9. *$D_s \partial_t \Gamma = D_t \partial_s \Gamma$, where $D_s = \frac{D}{ds}$ with respect to Γ_s.*

Proof. Say $\Gamma = (f^1(s,t), \dots, f^n(s,t))$ locally. Then

$$\partial_t \Gamma = \sum_k \frac{\partial f^k}{\partial t} \frac{\partial}{\partial x^k}, \qquad \partial_s \Gamma = \sum_k \frac{\partial f^k}{\partial s} \frac{\partial}{\partial x^k},$$

$$D_s \partial_t \Gamma = \sum_k \left(\frac{\partial^2 f^k}{\partial s \partial t} + \sum_{i,j} \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial s} \Gamma^k_{i,j} \right) \frac{\partial}{\partial x^k}$$

$$D_t \partial_s \Gamma = \sum_k \left(\frac{\partial^2 f^k}{\partial t \partial s} + \sum_{i,j} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial t} \Gamma^k_{i,j} \right) \frac{\partial}{\partial x^k}$$

So we have to show that $\Gamma^k_{ij} = \Gamma^k_{ji}$. By definition, $\nabla_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial x^j}) = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$ and by symmetry of ∇

$$\nabla_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial x^j}) - \nabla_{\frac{\partial}{\partial x^j}}(\frac{\partial}{\partial x^i}) = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0.$$

Thus $\Gamma^k_{ij} = \Gamma^k_{ji}$. □

Definition 16.10. Fix a point p and consider $\exp\colon T_pM \to M$. Let $B_R(0) = \{v \in T_pM : |v| < R\}$ and $S_R(0) = \{v \in T_PM : |V| = R\}$. A *geodesic ball* of radius R around p is $\exp(B_R(0))$ and a *geodesic sphere* around of radius R around p is $\exp(S_R(0))$.

Theorem 16.11 (Gauß' lemma)**.** *Let U be a geodesic ball around p. Then $\frac{\partial}{\partial r}$ is orthogonal to geodesic spheres in $U \setminus \{p\}$ (here r is with respect to normal coordinates).*

Proof. Fix $q \neq p$ in U and let $v \in T_pM$ with $q = \exp(v)$. Further let $X \in T_qM$ be tangential to the geodesic sphere through q. We have to show that X and $\frac{\partial}{\partial r}\big|_q$ are orthogonal. The geodesic from p to q is given by $\gamma(t) = \exp(tv)$, $t \in [0,1]$, which is radial. So $\dot\gamma(t)$ is proportional to $\frac{\partial}{\partial r}$. We will show that $\dot\gamma(1)$ is orthogonal to X.

Pick $w \in T_qM$ such that $(D\exp)\colon T_vT_pM \cong T_pM \to T_qM$ has $D\exp(w) = X$. Pick a curve $\sigma \in T_pM$ with $\sigma(0) = v$, $\dot\sigma(0) = w$ and $|\sigma| \equiv R$ (this is possible as $|v| = R$ and $\exp(w) = X$ is tangential to the geodesic sphere with radius R. Set

$$\Gamma(s,t) = \exp(t\sigma(s)), \quad S = \partial_s \Gamma, \quad T = \partial_t \Gamma.$$

Then

$$\begin{aligned} &S(0,0) = 0 \text{ as } \Gamma_0(s) \equiv p; \\ &S(0,1) = \frac{d}{ds}\bigg|_{s=0} \exp(\sigma(s)) = D\exp(\sigma(0)) = D\exp(w) = X; \\ &T(0,0) = v \text{ as } \Gamma(0,t) = \gamma(t); \\ &T(0,1) = \dot\gamma(1). \end{aligned}$$

Trivially, $S(0,0)$ is orthogonal to $T(0,0)$. We want to show that $S(0,1)$ is orthogonal to $T(0,1)$. Note that $D_t \partial_t \Gamma = 0$ since as $\partial_t \Gamma$ is parallel as Γ is given by a geodesic for fixed s.

$$\begin{aligned}\frac{d}{dt}\langle S,T\rangle &= \langle D_tS,T\rangle + \langle S, D_tT\rangle\\ &= \langle D_t\partial_s\Gamma, T\rangle + \langle S, D_t\partial_t\Gamma\rangle\\ &= \langle D_t\partial_s\Gamma, T\rangle = \langle D_sT, T\rangle\\ &= \frac{1}{2}\frac{d}{ds}\langle T,T\rangle = \frac{1}{2}\frac{d}{ds}|T| = \frac{1}{2}\frac{d}{ds}tR = 0\end{aligned}$$

So $\langle S,T\rangle = \text{constant} = 0$ and hence X is orthogonal to $\dot\gamma(1)$. □

Theorem 16.12. *Let $x^1,\dots,x^n$ be normal coordinates centered at p on $U \subseteq M$. For any point $q \in U$, the radial geodesic from p to q is (up to reparametrization) the unique curve of shortest length from p to q.*

Proof. Let $U = \exp(B_R(0))$ and let $v \in T_pM$ with $q = \exp(v)$. Set $\varepsilon = |v| < R$. Then $\gamma(v) = \exp(tv) = (tv^1,\dots,tv^n)$ is the radial geodesic from p to q.

$$l(\gamma) = \int_0^1 |\dot\gamma(t)|dt = \varepsilon.$$

Let $\sigma\colon [0,1] \to U$ with $\sigma(0) = p$ and $\sigma(1) = q$. First, we want to show that $l(\sigma) \geq \varepsilon$. Write $\dot\sigma = f(t)\frac{\partial}{\partial r} + Z(t)$ with $Z(t)$ tangent to the geodesic sphere through $\sigma(t)$. By Gauß' lemma, $|\dot\sigma(t)|^2 = f(t)^2 + |Z(t)|^2 \geq f(t)^2$ as $|\frac{\partial}{\partial r}| = 1$. Also $\frac{d}{dt}r(\sigma(t)) = \dot\sigma(t)(r) = f(t) + \underbrace{Z(t)(r)}_{=0}$. So

$$l(\sigma) = \int_0^1 |\dot\sigma(t)|dt \geq \int_0^1 f(t)dt = \int_0^1 \frac{d}{dt}r(\sigma(t)) = r(\underbrace{\sigma(1)}_{q}) - r(\underbrace{\sigma(0)}_{p}) = \varepsilon.$$

Now, if $l(\sigma) = \varepsilon$, we must have $Z(t) \equiv 0$, i.e. $\dot\sigma(t) = f(t)\frac{\partial}{\partial r}$. If we reparametrize σ to have unit speed, $f(t) \equiv 1$, then σ is a radial geodesic. □

Definition 16.13. A curve γ is *locally minimizing* if whenever $p = \gamma(t_0)$ for some t_0, there exists a neighborhood U of p such that if $q_i = \gamma(t_i) \in U$ $(i = 1,2)$, then γ is the shortest path from q_1 to q_2.

Theorem 16.14. *A curve γ is a geodesic if and only if it is locally minimizing.*

Proof. handout (non-examinable). □

17 Curvature of Riemannian manifolds

Let M be a manifold, g a Riemannian metric and ∇ a Levi-Civita connection. Recall that the curvature is represented by a form $R \in \Omega^2(\mathrm{End}(TM))$. So for $X, Y \in \mathrm{Vect}(M)$, we have $R(X,Y) \in \Omega^0(\mathrm{End}(TM))$ given by $\mathcal{R}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. Given a third vector field Z, we set

$$\mathcal{R}(X,Y)Z = \nabla_X\nabla_YZ - \nabla_Y\nabla_XZ - \nabla_{[X,Y]}Z.$$

This is a tensor of type $(3,1)$:

$$\begin{aligned}\mathcal{R}(X,fY)Z &= \nabla_Xf\nabla_YZ - f\nabla_Y\nabla_XZ - \nabla_{[X,fY]}Z =\\ &\qquad X(f)\nabla_YZ + f\nabla_X\nabla_YZ - f\nabla_Y\nabla_XZ - f\nabla_{[X,Y]}Z - X(f)\nabla_YZ = f\mathcal{R}(X,Y)Z.\end{aligned}$$

Similarly it is $C^\infty(M)$-linear in the other variables. If $x^1, \ldots, x^n$ are coordinates,

$$R = \sum_{i,j,k,l} R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l} \qquad \text{with}$$

$$\sum_l R^l_{ijk} \frac{\partial}{\partial x^l} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k}.$$

Definition 17.1. The *Riemannian tensor* is $R_m(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle_g$.

It can be written as

$$R_m = \sum R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l, \qquad R_{ijkl} = \sum_m g_{lm} R^m_{ijk}.$$

Lemma 17.2. *The Riemannian tensor has the following symmetries:*

1. $R_m(X, Y, Z, W) = -R_m(Y, X, Z, W)$;
2. $R_m(X, Y, Z, W) = -R_m(X, Y, W, Z)$;
3. $R_m(X, Y, Z, W) = R_m(W, Z, X, Y)$;
4. $R_m(X, Y, Z, W) + R_m(Y, Z, X, W) + R_m(Z, X, Y, W) = 0$;

These say

$$R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} = -R_{klij}, \quad R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$

Definition 17.3. A manifold M with Riemannian metric g is called *flat* if $\mathcal{R} \equiv 0$.

Theorem 17.4. *A manifold M with Riemannian metric g is called* flat *if and only if it is locally isometric to $\mathbb{R}^n$, i.e. if around each point $p \in M$ there exists a chart (x, U) such that Dx is an isometry (where $\mathbb{R}^n$ is endowed with the usual metric $\sum_{i,j} \delta_{ij} dx_i \otimes dx_j$) or equivalently if there exist local coordinates such that $g = \sum_{i,j} \delta_{i,j} dy^i \otimes dy^j$ on U.*

List of Notation

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